2012 Junior Division Second Round Solution

1. 【**Solution 1**】Since we can get a second cup of juice by paying 1 more dollar while buying a cup at the regular price of 7 dollars, we will get 8 cups by buying 4 cups and paying 4 more dollars. There are 9 persons, so we need 1 more cup. They totally cost 4 × 7 + 4 + 7 = 39 dollars.
【**Solution 2**】We know that we can get 1 more cup of juice by paying 1 more dollar while buying a cup at the price of 7 dollars. That is, 2 cups of juice cost at least 8 dollars. Now we need 9 cups. Since 9 = 2 × 4 + 1, it cost us at least 4 × 8 + 1 × 7 = 39 dollars to buy the drink.

Answer：（C）

2. Notice that $\frac{x^2}{1+x^2}+\frac{\left(\frac{1}{x}\right)^2}{1+\left(\frac{1}{x}\right)^2}=\frac{x^2}{1+x^2}+\frac{1}{1+x^2}=1$, then

$$\frac{4^2}{1+4^2}+\frac{\left(\frac{1}{4}\right)^2}{1+\left(\frac{1}{4}\right)^2}=1 、\frac{8^2}{1+8^2}+\frac{\left(\frac{1}{8}\right)^2}{1+\left(\frac{1}{8}\right)^2}=1 、\cdots 、\frac{2012^2}{1+2012^2}+\frac{\left(\frac{1}{2012}\right)^2}{1+\left(\frac{1}{2012}\right)^2}=1$$

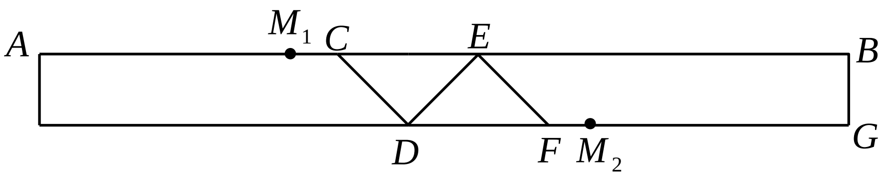
Hence, the result is 1 × 503 = 503.

Answer：(D)

3. Assume that there are x carnations and y roses in a bouquet. According to the problem, $3x + 4y = 60$, while x and y are positive integers. Since $0<3x<60$, $0<4y<60$, we know that $0<x<20$, $0<y<15$. Now we could rewrite the first formula into $4y=60-3x$. Both sides of the equation are divided by 4, so is $3x$. Since 3 isn't divided by 4, x is divided by 4. Therefore, $x = 4$, $x = 8$, $x = 12$ or $x = 16$, corresponded to $y = 12$, $y = 9$, $y = 6$ or $y = 3$. So there are 4 different kinds of bouquets.

Answer：(A)

4. Unfold the strip, the crease should show as the followed picture. By condition, $AM_1=GM_2$. Since $CM_1=FM_2=3$, we know that $AC = FG$. Hence, the picture is centrally symmetric. So $AC=\frac{30}{2}-3-\frac{3}{2}=10.5\text{cm}$.

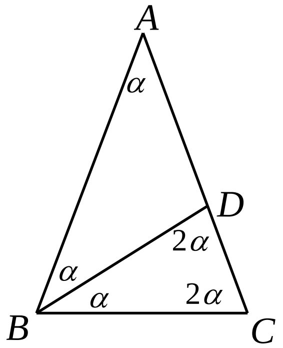


Answer：(B)

5. Rewrite $c=-\dfrac{ab}{a+b}$ into $ab+bc+ca=0$. In followed choices, only the difference of both sides in (B) is $(a+b+c)^2-(a^2+b^2+c^2)=2(ab+bc+ca)=0$. The difference of both sides in the other choices is not necessarily equal to 0. (For example, let $a=2$, $b=-1$, $c=2$. Hence, in (A), the left side is 3, the right side is 15; in (C), the left side is 25, the right side is -11; in (D), the left side is 27, the right side is 15; in (E), the left side is 27, the right side is 11.)

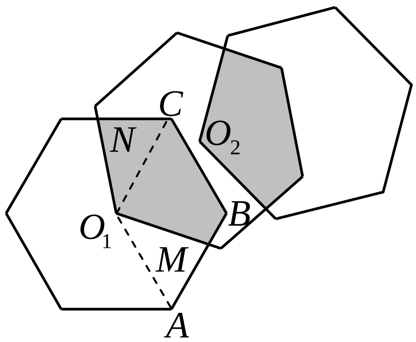
Answer：(B)

6. Assume that $\angle BAC=\alpha$. Since $AD = DB$, we know that $\angle ABD=\angle BAC=\alpha$. Therefore, $\angle BDC=2\alpha$. (Because the measure of an exterior angle of a triangle is equal to the sum of the measures of the two interior angles that are not adjacent to it.) Because $DB = BC$, $\angle BCD=\angle BDC=2\alpha$; $AB = AC$, $\angle ABC=\angle ACB=2\alpha$. In $\triangle ABC$, three internal angles are α, 2α, 2α. Hence, $\alpha+2\alpha+2\alpha=180°$, which leads to $\alpha=36°$.



Answer：36 degrees

7.



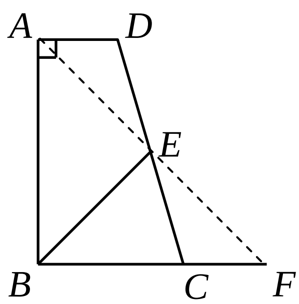
First, we try to find out the area of one shaded part. As shown, the pentagon O_1MBCN is the left shaded part of the original problem. From the property of regular hexagon, $\angle MO_1N=120°=\angle AO_1C$. Eliminate the overlapping angle, we know $\angle MO_1A=\angle NO_1C$. Notice that $O_1A=O_1C$, $\angle MAO_1=\angle NCO_1=60°$ (A property of regular hexagon). Therefore, $\Delta MO_1A\cong\Delta NO_1C$, $S_{\Delta MO_1A}=S_{\Delta NO_1C}$. That is, the area of O_1MBCN equals to the area of O_1ABC, while the latter is one-third of the area of a regular hexagon. Hence, the area of the left shaded part is $12\times\dfrac{1}{3}=4$ cm^2. Similarly, the area of the right shaded part is 4 cm^2, too. So the area of shaded part is totally 8 cm^2.

Answer：8 cm^2

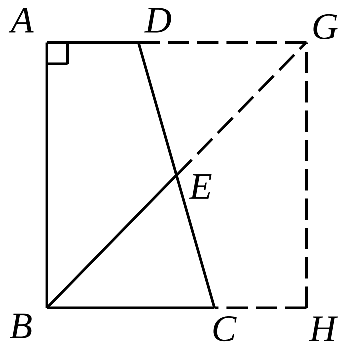
8. By condition, $2013^{\frac{1}{a}}=(3^a)^{\frac{1}{a}}=3^{a\times\frac{1}{a}}=3$ 、 $2013^{\frac{1}{b}}=(671^b)^{\frac{1}{b}}=671^{b\times\frac{1}{b}}=671$. Hence, $2013^{\frac{1}{a}+\frac{1}{b}}=3\times671=2013=2013^1$, which leads to $\dfrac{1}{a}+\dfrac{1}{b}=1$.

Answer：1

9. 【**Solution 1**】 Connect *AE* and lengthen it, which meets *BC* at *F*. Because *E* is the midpoint of *CD*, *AD* // *BC*, $\Delta ADE \cong \Delta FCE$. Hence, *AD* = *CF*, *AE* = *FE*. Also, *AB* = *AD* + *BC* = *CF* + *BC* = *BF*, $AB \perp BF$, so $\triangle ABF$ is an isosceles right triangle. Therefore, *AE* = *EF* = *BE* = 20, $BE \perp AF$. So the area of $\triangle ABF$ is $\frac{1}{2}\times 20\times(20+20)$ $=400$. Due to $\triangle ADE \cong \triangle FCE$, the area of *ABCD* is equal to that of $\triangle ABF$, so the answer is 400.



【**Solution 2**】Lengthen *AD* to *G*, where *GD*=*BC*; lengthen *BC* to *H*, where *CH*=*AD*. Since *AB* = *AD* + *BC* = *AG* = *BH*, $AB \perp BF$, *ABHG* is a square, and the area of *ABCD* is half of that of *ABHG*. Because *E* is the midpoint of *CD*, *E* is also the midpoint of *BG*. That is, the length of *ABHG*'s diagonal, *BG*, is 40. So the area of *ABHG* is $40\times 40\div 2=800$. Therefore, the area of *ABCD* is 400.



Answer：400

10. We find that 792 = 8 × 9 × 11, and the greatest common divisor of 8, 9 and 11 is 1, so $\overline{20ab13c}$ should be divided by 8, 9 and 11. From the property of numbers divided by 8, we know that 130 + *c* is divided by 8, implying that *c* = 6; from the property of numbers divided by 9, we know that $2+0+a+b+1+3+6$ is divided by 9, implying that $a+b=6$ or $a+b=15$; from the property of numbers divided by 11, we know that $a-b=5$ or $a-b=-6$. Notice that $a+b$ and $a-b$ are either odd or even. Consider $a+b=15$ and $a-b=5$, we know that *a*=10, *b*=5, which is a contradiction; consider $a+b=6$ and $a-b=-6$, we know that *a*=0, *b*=6. Hence, *c* = 6. So, $c(a+b)=36$.

Answer：36

11. Assume that there are *x* men in the party. Since each man handshakes with 4 women, the number of handshakes between men and women is 4*x*. Also, each woman handshakes with 6 men, so there are $\frac{4x}{6}=\frac{2}{3}x$ women. Because each man handshakes with 6 men, the number of handshakes between men is $\frac{1}{2}\times 6x=3x$. Each woman handshakes with 4 women, so the number of handshakes between women is $\frac{1}{2}\times 4\times\frac{2}{3}x=\frac{4}{3}x$. We know that $3x+\frac{4}{3}x-4x=7$, that is, *x*=21.

Answer：21

12. The number of balls in each box can only be one of 10, 11, …, 20. Consider that we put away one or several boxes from boxes containing 10, 11, …, 20 balls, making the number of the rest of balls is 130. Since 10+11+12+…+20－130=35, we should put away 35 balls. From $20\times1<35<10\times4$, we should remove 2 or 3 boxes. If we remove 2 boxes, the balls might be either {15, 20}, {16, 19}, or {17, 18}, totally 3 possibilities; if we remove 3 boxes, the balls might be either {10, 11, 14} or {10, 12, 13}, totally 2 possibilities. Hence, there are 5 different ways.

Answer：5 ways

13. Assume that $a-2b=p$, while *p* is a prime. There are two different cases:
(Case 1) If *a* is divided by *p*, 2*b* is also divided by *p*. Let $2b=p\cdot r$, $a=p\cdot(r+1)$, where *r* is a non-negative integer. Hence, $2ab=p^2r(r+1)$ is a perfect square number. That is, $r(r+1)$ is a perfect square number. Since the product of two consecutive positive integer isn't a perfect square, *r* should be 0. Then, $b=0$, $a=p$ and $a+b=p$. Since the largest prime smaller than 100 is 97, the maximum value of $a+b$ is 97.
(Case 2) If *a* is not divided by *p*, 2*b* is not divided by *p*, either. Therefore, $(a,2b)=(p,2b)=1$. Because $2ab=a\cdot 2b$ is a perfect square number, *a* and 2*b* are both perfect square numbers. Let $a=m^2$, $2b=n^2$, then $a-2b=m^2-n^2=(m+n)(m-n)$ is a prime. So, $m-n$ can only be 1, that is, $m=n+1$. Then $m+n=2n+1$ is a prime. Because $2b=n^2<100$, *n* is an even not larger then 8. In consequence, *n* should be 2, 6 or 8. While *n* = 2, *a* = 9, *b* = 2, $a+b=11$; while *n* = 6, *a* = 49, *b* =18, $a+b=67$; while *n* = 8, *a* = 81, *b* = 32, $a+b=113$. The maximum value of $a+b$ is 113.
To sum up, the maximum value of $a+b$ is 113.

Answer：113

14. Since $x^4+2x^3+(3+k)x^2+(2+k)x+2k=(x^2+x+2)(x^2+x+k)$, (5 marks) $x^2+x+2=0$ has no real roots, all real roots of the equation $x^4+2x^3+(3+k)x^2+(2+k)x+2k=0$ are all real roots of the equation $x^2+x+k=0$. (5 marks)
By condition we know that $x^2+x+k=0$ has real roots, and the product of the roots is -2012. By Vieta's formulas, we know that $k=-2012$. Let the real roots of equation $x^2+x-2012=0$ are x_1 and x_2. Then $x_1+x_2=-1$, $x_1x_2=-2012$ (5 marks)
Hence,

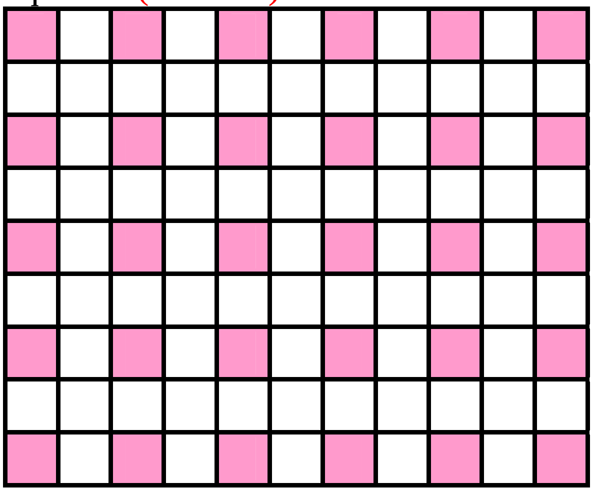
$$x_1^2+x_2^2=(x_1+x_2)^2-2x_1x_2=1^2-2\times(-2012)=4025.$$ (5 marks)

Answer：4025

【Marking Scheme】
Factorizing the original equation exactly, 5 marks;
Knowing that all real roots of the original equation are all real roots of

$x^2 + x + k = 0$, 5 marks;
Using the Vieta's theorem exactly, 5 marks;
Using the formula of sum of perfect squares to find the sum of squares of real roots exactly, 5 marks;
Only exact solution without the solving process, 5 marks.

15. As shown, we color the squares lie on both odd row and odd column. There are totally 5×6=30 red squares. (5 marks)



Generally call these shapes "polyomino", specifically called the first shape "tromino".

It's clear that a polyomino cover at most a red square. Hence, we need at least 30 polyominoes to cover the chessboard. Assume that m trominoes are used, n copies of other two shapes are used, then

$$m + n \geq 30.$$ (5 marks)

Also, each tromino will cover 3 squares, each copy of the other two shapes will cover 4 squares, then

$$3m + 4n = 9 \times 11 = 99.$$

Hence,

$$4n = 99 - 3m.$$

Also,

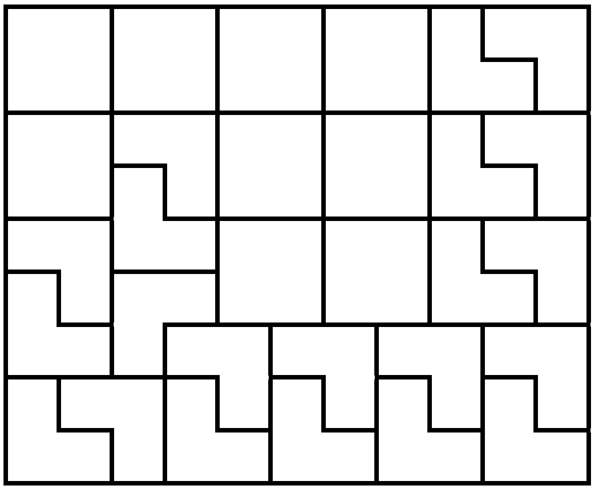
$$4m + 4n \geq 120,$$

Therefore,

$$4m + (99 - 3m) \geq 120,$$

$$m \geq 21.$$ (5 marks)

So we need at least 21 trominoes. As followed is a case satisfied. (5 marks)



Answer：21 copies

【Marking Scheme】

Giving the exactly painting way , 5 marks;

Knowing $m+n\geq 30$, 5 marks;

Find $m\geq 21$, 5 marks;

Construct a covering way satisfied the conditions, 5 marks;

Only numerical solution without constructing a covering way, 0 marks.