Solution Key to Second Round of IMAS 2015/2016 Junior High School Division

1. Michael bought 6 pens and 3 notebooks while Wallace bought 3 pens and 6 notebooks. The pens are identical and so are the notebooks. Michael's bill is 6 dollars higher than Wallace's. How many dollars is the price of a pen higher than the price of a notebook?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

【Suggested Solution】

Since 3 pens cost 6 dollars more than 3 notebooks, a pen costs 2 dollars more than a notebook.

Answer: (B)

2. If all the divisors of 2016 are arranged in decreasing order, by how much is the third divisor larger than the fourth divisor?

   (A) 12 (B) 48 (C) 168 (D) 672 (E) 2016

【Suggested Solution】

Since 2016 is divisible by 2, 3 and 4, its third largest divisor is $2016 \div 3 = 672$ and its fourth largest divisor is $2016 \div 4 = 504$. Their difference is $672 - 504 = 168$.

Answer: (C)

3. Which of the following is equal to $xr + ys$ if $r = 3x + 2y$ and $s = xy - x - y$?

   (A) $x^2y - x^2 + 2xy + 2y^2$ (B) $xy^2 + 3x^2 + xy - y^2$ (C) $x^2y + 2x^2 + xy$

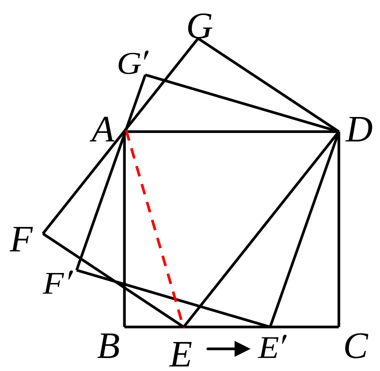
   (D) $xy^2 + 2x^2 + 2xy$ (E) $x^2y^2 + x + y$

【Suggested Solution】

$xr + ys = x(3x + 2y) + y(xy - x - y) = 3x^2 + 2xy + xy^2 - xy - y^2 = xy^2 + 3x^2 + xy - y^2$。

Answer: (B)

4. E is a variable point on the side BC of a square $ABCD$. $DEFG$ is a rectangle with FG passing through A. As the point E moves from B towards C, how does the area of $DEFG$ change?



   (A) Steadily increasing (B) Steadily decreasing

   (C) Decreasing and then increasing (D) Increasing and then decreasing

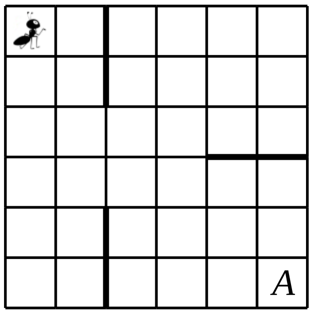
   (E) Remaining constant

【Suggested Solution】

The area of triangle *ADE* is constant since it has a fixed base *AD* and a constant altitude. The area of *DEFG* is always twice the area of *ADE* since they have a common base *DE* and the same altitude.

Answer: （E）

5. The diagram shows a 6 by 6 board with three barriers. An ant is at the top left corner and wishes to reach the bottom right corner. It may only crawl between squares which share a common side, and only towards the bottom or the right. It cannot pass through any barrier. How many different paths can it follow?



（A）88 （B）90 （C）92 （D）96 （E）112

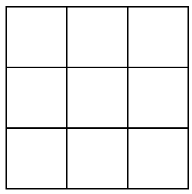
【Suggested Solution】

The diagram below shows the number of ways the ant can reach each square on the board according to the rules.

1	1	0	0	0	0
1	2	0	0	0	0
1	3	3	3	3	3
1	4	7	10	10	10
0	0	7	17	27	37
0	0	7	24	51	88

Answer: （A）

6. One square in a 3 by 3 board is to be painted black, a second square blue and a third square red. If no two of these three squares are in the same row or in the same column, how many different ways of painting them are there?

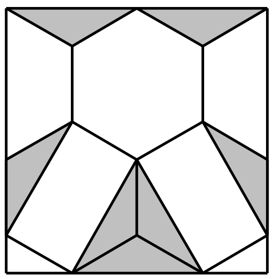


【Suggested Solution】

There are 9 possible places for the black square, 4 for the blue square and only 1 for the red square. The total number of choices is $9\times4\times1=36$.

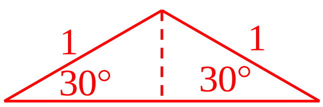
Answer: 36

7. The diagram shows a tile divided into regular hexagons of side length 1 cm. What is the total area, in cm^2, of the parts of the tile which are shaded?



【Suggested Solution】

The shaded parts consist of seven triangles of sides 1, 1 and $\sqrt{3}$.



The altitude is $\frac{1}{2}$ and the area is thus $\frac{1}{2}\times\frac{1}{2}\times\sqrt{3}=\frac{\sqrt{3}}{4}$. The total area of the shaded parts is therefore $\dfrac{\sqrt{3}}{4}\times 7=\dfrac{7\sqrt{3}}{4}$.

Answer: $\dfrac{7\sqrt{3}}{4}\text{cm}^2$

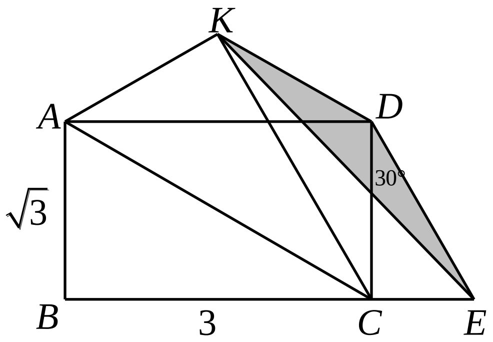
8. Let a, b, c and d be real numbers such that $|a+b|$, $|a-b|$, $|c+d|$ and $|c-d|$ are 6, 7, 8 and 9 in some order. What is the value of $a^2+b^2+c^2+d^2$?

【Suggested Solution】

From $(a+b)^2=(|a+b|)^2=6^2=36$ and $(a-b)^2=(|a-b|)^2=7^2=49$, we have $a^2+b^2=\dfrac{1}{2}((a+b)^2+(a-b)^2)=\dfrac{1}{2}\times(36+49)$. Similarly, $c^2+d^2=\dfrac{1}{2}((c+d)^2+(c-d)^2)=\dfrac{1}{2}\times(64+81)$. It follows that $a^2+b^2+c^2+d^2=\dfrac{1}{2}\times(36+49)+\dfrac{1}{2}\times(64+81)=\dfrac{1}{2}\times 230=115$.

Answer: 115

9. A rectangle *ABCD* with $BC=3\text{cm}$ and $AB=\sqrt{3}\,\text{cm}$ is folded along *AC* so that the point *B* lands on the point *K* symmetric to it about *AC*. What is the area, in cm^2, of triangle *KDE*, where *E* is the point on the extension of *BC* such that $\angle CDE=30°$?



【Suggested Solution】

Since $CD=AB=\sqrt{3}\,\text{cm}$ and $\angle CDE=30°$, we have $DE=2\text{cm}$;

From $AB:BC=\sqrt{3}:3=1:\sqrt{3}$ and $\angle ABC=90°$, $\angle ACB=30°$. Hence $\angle ACK=30°$, so that $\angle DCK=90°-30°-30°=30°$.

It follows that *KC* is parallel to *DE*, so that the area of *KCD* is the same as that of *CDE*, which is $\dfrac{1}{2}\times 2\times\dfrac{\sqrt{3}}{2}=\dfrac{\sqrt{3}}{2}\text{cm}^2$

Answer: $\dfrac{\sqrt{3}}{2}\text{cm}^2$

10. In the expression $(((10\square 2)\square 2)\square 2)\square 2$, each $\square$ is to be replaced by a different one of +, −, × and ÷. How many different values can this expression take?

【Suggested Solution】

If the addition and the subtraction are performed consecutively, they will cancel out. The same applies to the multiplication and the division. To obtain a value other than 10, we must alternate the first pair with the second. Thus we only have to examine eight cases:

$(((10+2)\times 2)-2)\div 2=11$ 、 $(((10+2)\div 2)-2)\times 2=8$ 、 $(((10-2)\times 2)+2)\div 2=9$ 、 $(((10-2)\div 2)+2)\times 2=12$ 、 $(((10\times 2)-2)\div 2)+2=11$ 、 $(((10\times 2)+2)\div 2)-2=9$ 、 $(((10\div 2)-2)\times 2)+2=8$ 、 $(((10\div 2)+2)\times 2)-2=12$.

Altogether, there are five possible values, namely, 8, 9, 10, 11 and 12.

Answer: 5

11. Let a, b and c be real numbers such that $abc=1$, $(a+1)(b+1)(c+1)=16$ and $(a+2)(b+2)(c+2)=53$. What is the value of $(a-1)(b-1)(c-1)$?

【Suggested Solution】

From $(a+1)(b+1)(c+1)=abc+ab+ac+bc+a+b+c+1$, we have $ab+ac+bc+a+b+c=16-abc-1=14$. From $(a+2)(b+2)(c+2)=abc+2ab+2ac+2bc+4a+4b+4c+8$, we have

$$ab+ac+bc+2a+2b+2c=\frac{53-abc-8}{2}=22\ ;$$

Subtraction yields $a+b+c=22-14=8$. Hence $ab+ac+bc=14-8=6$.

It follows that $(a-1)(b-1)(c-1)=abc-ab-ac-bc+a+b+c-1=1-6+8-1=2$.

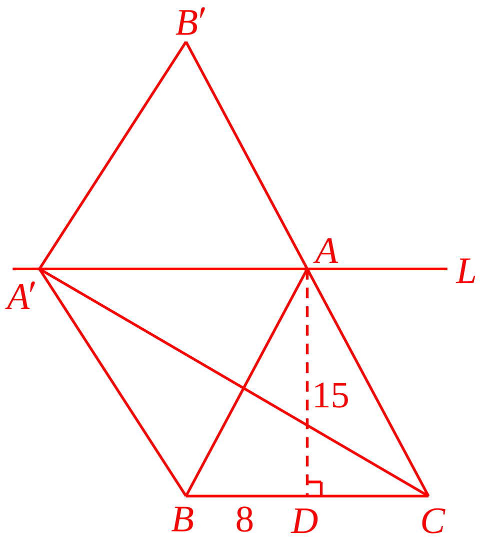
Answer: 2

12. The area of triangle ABC is 120 cm^2 and $BC=16\,\text{cm}$. What is the minimum length, in cm, of the perimeter of ABC?

【Suggested Solution】

The altitude of ABC to the base BC has length $\frac{120\times 2}{16}=15$. When $AB=AC$, we have $AC=AB=\sqrt{15^2+8^2}=17$ so that the perimeter of ABC is 16+17+17=50. We claim that this is minimum. Let B' be the reflection of B across the line through A parallel to BC. Then A lies on CB' and $B'C=AB'+AC=AB+AC$. For any other point A' on this line,

$$A'B+A'C=A'B'+A'C>B'C=AB+AC$$



Answer: 50 cm

13. Let a, b, c and d be four different non-zero digits. The greatest common divisor of the four-digit numbers $\overline{abcd}$ and $\overline{acbd}$ is n. What is the largest possible value of n?

【Suggested Solution】

We may assume by symmetry that $b > c$. Then $\overline{abcd} - \overline{acbd} = 90(b-c)$, which is divisible by the desired greatest common divisor n. Since $d \neq 0$, n is not divisible by 10. If n is not divisible by 5, then it is at most $\frac{90(b-c)}{5} = 18(b-c) \geq 18 \times 8 = 144$. If n is divisible by 5, then it is a divisor of $\frac{90(b-c)}{2} = 45(b-c)$. Since $b-c \leq 9-1=8$, $n \leq 45\times 7 = 315$. If $n = 315$,then d=5 and (b,c)=(9,2) or (8,1). Since 315 is divisible by 9, so is the digit-sum of $\overline{abcd}$, which is therefore either 4815 or 2925. However, neither is divisible by 7. The next highest value for n is $45\times 5 = 225$. As before, d=5. If we take b=7 and c=2, then both 4725 and 4275 are divisible by 225. Hence this is the maximum value for n.

Answer: 225

14. The first diagram shows a 6 by 6 board, and the second diagram shows an L-shaped piece consisting of four 1 by 1 squares. Paint as few of the squares of the 6 by 6 board black so that wherever the L-pieced piece is placed on the board covering four squares, at least one of the squares will be black. The L-shaped piece may be turned about or flipped over.

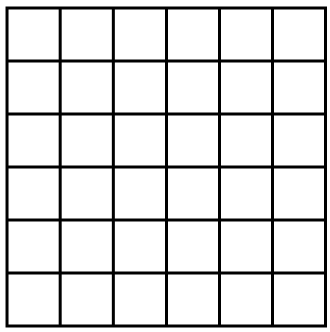


Figure 1

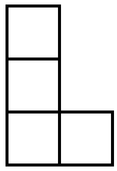
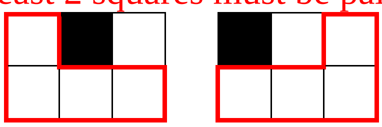
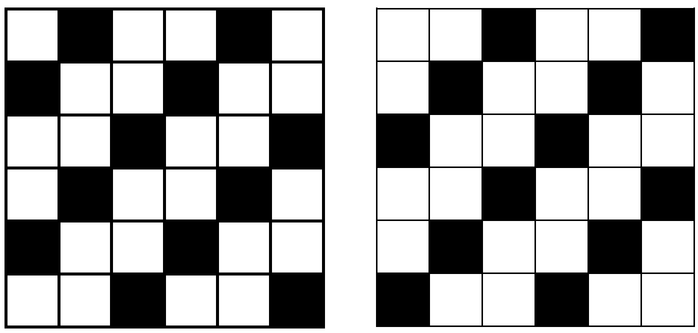


Figure 2

【Suggested Solution】

If we paint black all squares on three parallel diagonals of respective lengths 3, 6, 3, any placement of the L-shaped pieces must cover one of these 12 squares. (10 points) This is in fact minimum since the board may be divided into six 2 by 3 subboards, and within each subboard, at least 2 squares must be painted black. (10 points)





Answer: 12

15. P and Q are points on the bisector of the exterior angle at A of triangle ABC, with A between P and Q, such that BP is parallel to CQ. D is the point on BC such that $DP=DQ$. Prove that AB is parallel to DQ.

【Suggested Solution】

We first assume that $AB>AC$, so that the extensions of PQ and BC meet at some point X. Let E be the point on BC such that QE is parallel to AB. Then we have $\frac{XQ}{XA}=\frac{XE}{XB}$ and $\frac{XQ}{XP}=\frac{XC}{XB}$, that is $XA\times XE=XB\times XQ=XP\times XC$, hence $\frac{XA}{XP}=\frac{XC}{XE}$, so that PE is parallel to AC. (10 points)

Now $\angle EPQ=\angle CAQ=\angle PAB=\angle EQP$, so that $EP=EQ$. It follows that E coincides with D, and DQ is parallel to AB. (5 points)

The case $AB=AC$ is similar. (5 points)