Solution Key to Second Round of IMAS 2016/2017
Upper Primary Division

1. What is the value of $44\times 49\times 25$?
   (A) 43900 (B) 52900 (C) 53200 (D) 53825 (E) 53900

【Suggested Solution】

$44\times 49\times 25=11\times 49\times 25\times 4=539\times 100=53900$.

Answer：(E)

2. A hotel staff distributes four room keys to four travelers randomly. It is known that exactly two of the travelers get their correct room keys. How many different ways are there of distributing the keys to the four travelers?
   (A) 3 (B) 4 (C) 6 (D) 8 (E) 10

【Suggested Solution】

Exactly 2 of the 4 travelers get their correct keys, the number of ways is $\frac{4\times 3}{2}=6$.

Moreover, the 2 remaining travelers must not have got their correct keys, number of ways is 1. Therefore the total number of ways is $6\times 1=6$ ways.

Answer：(C)

3. How many factors of $3^4\times 5^6\times 7^{10}$ are relatively prime to 15?
   (A) 10 (B) 11 (C) 20 (D) 21 (E) 44

【Suggested Solution】

Since $15=3\times 5$, the only factors that are relatively prime to both $3^4\times 5^6\times 7^{10}$ and 15 are the powers of 7 (7^i, where i = 0, 1, 2, …, 10), which is a total of 11 numbers.

Answer：(B)

4. A total of 20 students and teachers visit a museum. The original price of each admission ticket is \$200. It is known that teachers are offered 10% discount, while students are offered 50% discount in purchasing the admission tickets. If the total admission fee is \$2640, how many teachers are there?
   (A) 6 (B) 8 (C) 10 (D) 14 (E) 16

【Suggested Solution 1】

From the conditions above, we can see that the price of each teacher ticket is $200\times 90\%=180$ dollars, and the price of each student ticket is $200\times 50\%=100$ dollars. Suppose that the number of teachers in the group is x, and then the number of students is $20-x$. Therefore, we have $180x+100(20-x)=2640$, we get $x=8$.

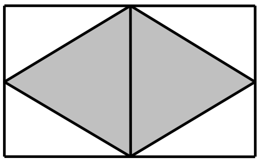
【Suggested Solution 2】

From the conditions above, we can see that the price of each teacher ticket is $200\times 90\%=180$ dollars, and the price of each student ticket is $200\times 50\%=100$ dollars, thus each student ticket is 80 dollars $(=180-100)$ cheaper than each teacher ticket. If all 20 people in the group are students, then the total cost should have been $20\times 100=2000$ dollars, which is 640 dollars $(=2640-2000)$ less than the actual amount paid. Therefore, the total number of teachers is $640\div 80=8$.

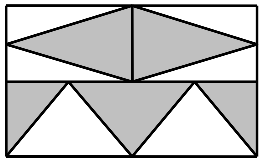
Answer：(B)

5. In each of the following options, the dimensions of the rectangle are 10 cm by 6 cm. There are some shaded triangles inside each rectangle. The vertices of the shaded triangles must be either at the endpoints of a line segment or points that divide it into equal parts. From the options below, which figure has the largest shaded region?

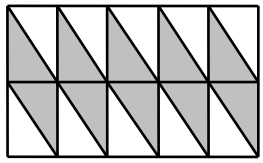
（A）



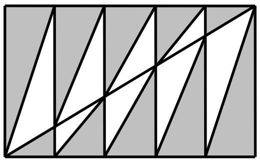
（B）



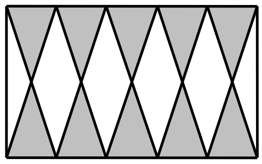
（C）



（D）



（E）



【Suggested Solution 1】

By finding the areas of the shaded triangles.

Shaded area of figure (A) is $2\times\frac{1}{2}\times(10\div2)\times6=30\,\text{cm}^2$.

Shaded area of figure (B) is

$2\times\frac{1}{2}\times(10\div2)\times(6\div2)+[10\times(6\div2)-2\times\frac{1}{2}\times(10\div2)\times(6\div2)]=30\,\text{cm}^2$.

Shaded area of figure (C) is $10\times\frac{1}{2}\times(10\div5)\times(6\div2)=30\,\text{cm}^2$.

Shaded area of figure (E) is $10\times\frac{1}{2}\times(6\div2)\times(10\div5)=30\,\text{cm}^2$.

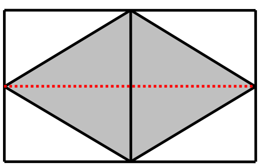
Note that areas of figures (A), (B), (C) and (E) are all equal. Now we examine figure (D). Note that the shaded parts of figure (D) are equally divided into two equal parts by a diagonal of the rectangle, with the sum of the areas equal. Now, observe the diagonals of the 5 small triangles on the bottom half of the rectangle. We can see that the bases of these five triangles are all equal to 2 cm, and their heights are 1.2 cm, 2.4 cm, 3.6 cm, 4.8 cm and 6 cm, respectively, from left to right which is in the ratio 1: 2: 3: 4: 5. Therefore, the sum of the area of the five shaded triangles is

$\frac{1}{2}\times(1.2+2.4+3.6+4.8+6)\times2=18\,\text{cm}^2$; therefore the total shaded area of the figure is 36 cm^2

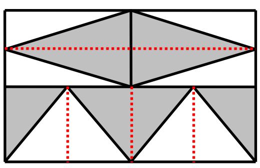
Therefore, the figure that has the largest shaded region is figure (D).

【Suggested Solution2】

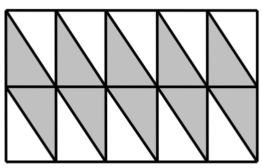
By drawing line segments into figures (A), (B), (C), and (E) below, observe that the shaded areas of each figure is equal to the half of the total area of the rectangle.



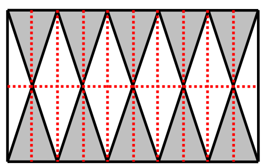
(A)



(B)

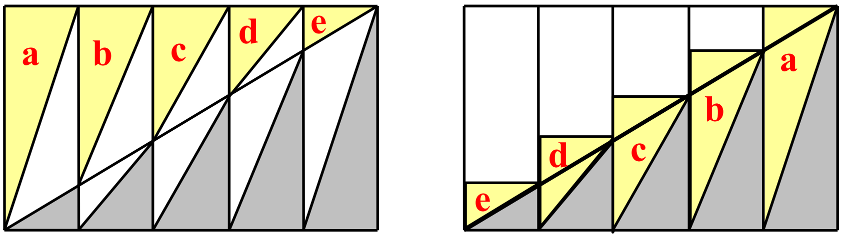


(C)



(D)

While in figure (D), observe that the sum of the area is greater than half of the entire rectangle.



Answer：(D)

6. Use the digits 2, 0, 1 and 7 once and without repetition to create all possible 4-digit numbers. How many of these numbers leave a remainder of 4 when divided by 11?

【Suggested Solution 1】

Using the divisibility rule of 11, where A is the sum of the digits of the thousands and tens digits, and B is the sum of the digits of the hundreds and units digits, we check the 6 different cases below:

	Thousands and Tens Digit	A	Hundreds and unit digit	$B-4$	Difference between A and $B-4$
(1)	7、2	$7+2$	1、0	$1+0-4$	1
(2)	7、1	$7+1$	2、0	$2+0-4$	1
(3)	7、0	$7+0$	2、1	$2+1-4$	8
(4)	2、1	$2+1$	7、0	$7+0-4$	0 or 11
(5)	2、0	$2+0$	7、1	$7+1-4$	2
(6)	1、0	$1+0$	7、2	$7+2-4$	4

Observe that only case (4) can satisfy the requirements, therefore, there are 4 numbers that satisfy the conditions namely 1027, 1720, 2017 and 2710.

【Suggested Solution 2】

Using the conditions above, we can from 18 different 4-digit numbers namely: 1027, 1072, 1207, 1270, 1702, 1720, 2017, 2071, 2107, 2170, 2701, 2710, 7012, 7021, 7102, 7120, 7201, 7210.

From the list, only 4 numbers will have a remainder of 4 when divided by 7 namely 1027, 1720, 2017, 2710.

Answer：4 numbers

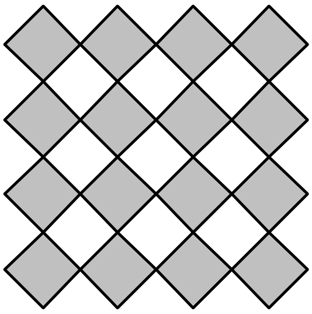
7. The lengths of two sides of a triangle are 6 cm and 13 cm respectively. It is known that the length of the third side in also an integer (in cm.). What is the minimum perimeter of this triangle?

【Suggested Solution】

From the triangle inequality, we can derive that the smallest possible side length of the third side is 8 cm, therefore, the minimum perimeter is $13+6+8=27$ cm.

Answer：27 cm

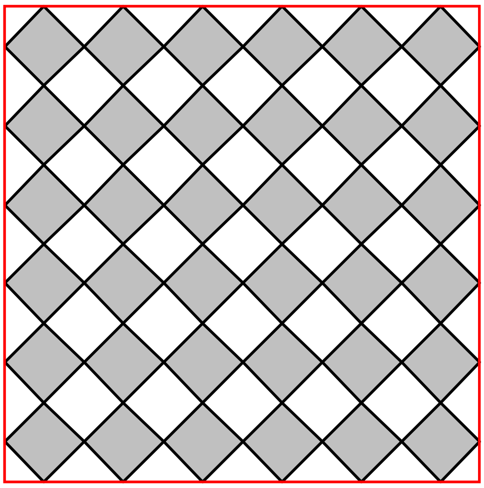
8. Refer to the diagram below, where 16 black squares and 9 white squares are arranged alternately such that it will form a figure in which there are 7 bricks on its diagonals, and all the outer bricks are black.



Using the similar pattern stated above, if we want to form figures in which there are 11 bricks on its diagonals, how many black square tiles are required?

【Suggested Solution 1】

Refer to the diagram on the right; you can see that the number of black square tiles used is $1+2+3+4+5+6+5+4+3+2+1=6^2=36$ black squares.



【Suggested Solution 2】

When there are 11 bricks on its diagonals, the number of lines taken is 6; with 6 black tiles on each line. Therefore, there are a total of $6\times6=36$ black squares used.

【Suggested Solution 3】

Enclose the figure with a big square as shown. Then the area of the regions covered by the white tiles is same as the that of the black tiles. Suppose each tile has side length equals 1. Then the area of the big square is $\frac{1}{2}\times12\times12=72$. Thus the number of the black tiles is $\frac{1}{2}\times72=36$.

Answer：36 black square tiles

9. Fill in the 4×4 box so that the numbers 1, 2, 3, and 4 appear exactly once in each row and column. Refer to the figure on the right, what is the sum of the values of A and B?

	A	4	
B		1	
1	2	3	4
3	4	2	1

【Suggested Solution】

Notice that the box that is situated below A can't be 1, 2, or 4, thus it must be 3, so $A=1$; while the box situated above box B can't be 1, 3 and 4, thus it must be 2, so we get $B=4$. So $A+B=1+4=5$.

Answer：5

2	1	4	3
4	3	1	2
1	2	3	4
3	4	2	1

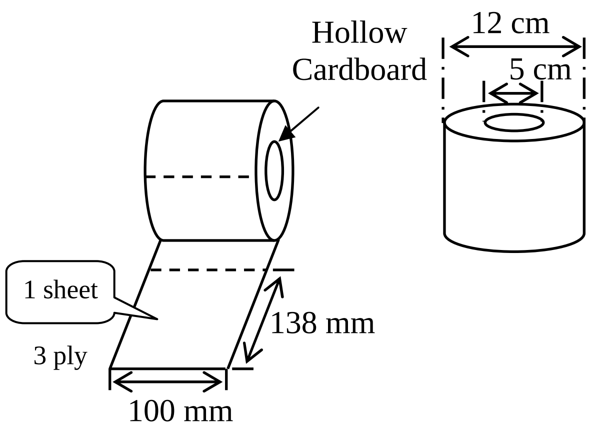
10. Alex and Charles both sent parcels of weight exceeding 10 kg . The postage rates Alex and Charles were both sending parcels. The postage rates are as follows: For the first 10 kg and below, the postage price is $6 per kg; the postage price for each successive kilogram after 10 kg is $2 lower than the original price. Given that Alex's parcel is 20% heavier than Charles' and his postage price is $12 more than that of Charles, find the weight, in kg, of the parcel sent by Alex?

【Suggested Solution】

Note that the package of Alex and Charles are both of weight heavier than 10 kg each. Also, Alex paid $12 more than Charles. Since the cost for each excess kg above 10 kg is $\$6-\$2=\$4$, the weight of Alex's package is $12\div 4=3$ kg heavier than Charles'. As Alexs' package is 20% heavier than Charles', Charles' package weighs $3\div 20\%=15$ kg and Alex's package weighs $15+3=18$ kg.

Answer：18 kg

11. Refer to the figure below where a pack of cylindrical-shaped tissue paper roll is shown, where the middle part is made up of a hollow rolled cardboard. The tissue paper is divided into several sheets and rolled over the cardboard. The pack indicates: "$138\text{ mm}\times 100\text{ mm}$ per sheet, 3 ply", which means that each sheet is 138 mm long and 100 mm wide and it contains 3 layers. Given that each roll of tissue paper is 0.13 mm thick, the diameter of cardboard roll is 5 cm, and the diameter of the rolled tissue paper is 12 cm, how many sheets of tissue paper, rounding to integer, are there in a pack? (Take $\pi=3.14$)



【Suggested Solution】

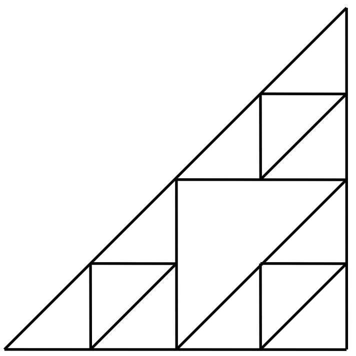
The volume of a cylinder is given by the product of its base area and its height. The height is 100 mm = 10 cm, so the volume of 1 roll is

$$3.14\times(\frac{12}{2})^2\times 10-3.14\times(\frac{5}{2})^2\times 10=934.15\text{ cm}^3 .$$

And since 138 mm = 13.8 cm, 0.13 mm = 0.013 cm, then the volume of each piece of toilet paper is $10\times 13.8\times 0.013\times 3=5.382\text{ cm}^3$. Therefore, the number of tissue paper sheets in 1 roll is $\frac{934.15}{5.382}\approx 173.57\approx 174$ sheets.

Answer：174 sheets

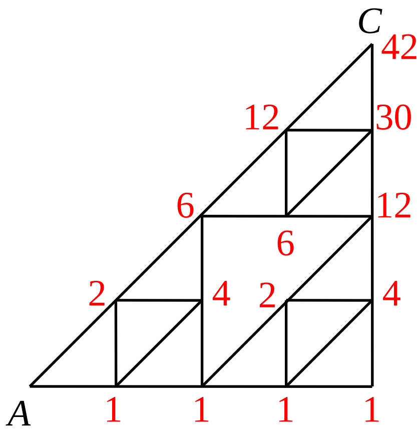
12. The diagram below is composed of many right angled isosceles triangles. Suppose an ant wants to travel from point A to point C, in how many ways can this be done if we are only allowed to move up, right or diagonally?



【Suggested Solution】

Refer to the diagram on the right, the total number of ways to go from A to C is 42 .

Answer：42 ways



13. From the set of positive integers from 1 to 174 inclusive, select 12 different positive integers such that their sum is 2017. In how many different ways can the 12 integers be selected?

【Suggested Solution】

Observe that if we add the 12 largest numbers, we get

$$174+173+172+171+170+169+168+167+166+165+164+163=2022,$$

which is 5 more than the required 2017.

Now we consider the cases below:

Case 1: We subtract 5 from one number.

And to avoid duplication, this number cannot be greater than or equal to 168, so there is a total of 5 ways in doing such:

$$174+173+172+171+170+169+168+167+166+165+164+158=2017$$

$$174+173+172+171+170+169+168+167+166+165+163+159=2017$$

$$174+173+172+171+170+169+168+167+166+164+163+160=2017$$

$$174+173+172+171+170+169+168+167+165+164+163+161=2017$$

$$174+173+172+171+170+169+168+166+165+164+163+162=2017$$

Case 2: We subtract 4 from one number and 1 from a different number.

We can only subtract 4 from the numbers less than 166 and subtract 1 from 163.

There are 2 possible ways:

(a) Subtract 4 from 165, and subtract 1 from 163:
$174+173+172+171+170+169+168+167+166+164+162+161=2017$

(b) Subtract 4 from 164, and subtract 1 from 163:
$174+173+172+171+170+169+168+167+166+165+162+160=2017$

Case 3: Subtracting not more than 3 from any number.
To subtract 3 from numbers less than 166 and subtracting 2 or 1 from numbers less than 165. No new solutions can be formed.
Therefore there is a total of $5+2=7$ ways.

Answer：7 ways

14. A bag contains 2017 balls that are numbered from 1 to 2017. At least how many balls must be taken out so that among those balls, there will always be 3 balls, in which the sum of the numbers on the first two balls is the number on the third ball?

【Suggested Solution】

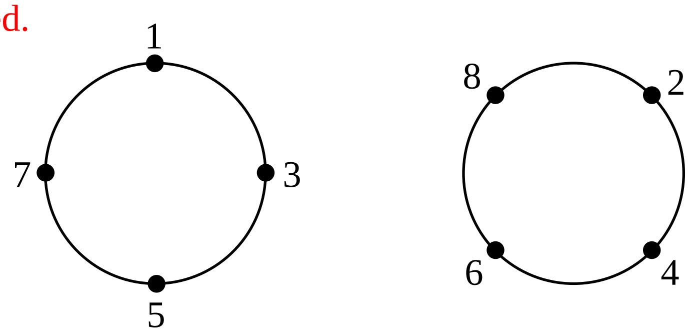
Since $2017=1008+1009<1009+1010$, suppose we take out 1009~2017, which is a total of 1009 numbers, we cannot ensure that there are three balls that will satisfy the given conditions.(10 points)
So suppose we take 1010 balls out. Suppose the largest number out of the 1010 balls is *M*, then the difference between *M* and the number of other balls taken out has 1009 different values, and both are less than 2017. (5 points) Since there are only 1007 balls that have not been taken out, at least one difference $M-x$ is the number y of the balls taken out, where *x* is the number of the removed ball. So *x*, *y*, $x+y=M$ are taken out of the ball number which satisfies the condition above. (5 points)

Answer：1010 balls

15. There are eight numbers namely 1, 2, 3, 4, 5, 6, 7 and 8. Choose at least two different numbers from the eight numbers such that the difference of any two of the chosen numbers neither equals to 2 nor 6. (For example, if 1 is already chosen, 3 and/or 7 cannot be chosen). What is the number of different ways to choose the numbers?

【Suggested Solution 1】

To understand the problem, we can consider to divide the eight numbers into two groups as shown in the diagram below. Observe that any two adjacent numbers in each group cannot be simultaneously selected, therefore, in each group, we can choose a maximum of two numbers only, and we can choose a maximum of four numbers combined.



Now we consider the 3 cases:

Case 1: If we choose 2 numbers:

Case 1a. If we get 1 number from each group, there is a total of $4 \times 4 = 16$ ways.

Case 1b. If we get 2 numbers from the same group, there is a total of $2 \times 2 = 4$ ways.

Therefore, we have a total of $16 + 4 = 20$ ways. (5 points)

Case 2: If we choose 3 numbers, we can get 2 numbers from 1 group, and 1 number from the other group, therefore we have a total of $2 \times 2 \times 4 = 16$ ways. (5 points)

Case 3: If we choose 4 numbers, we can get 2 numbers from both groups. Therefore, we have a total number of $2 \times 2 = 4$ ways. (5 points)

So, overall we have a total of $20 + 16 + 4 = 40$ ways. (5 points)

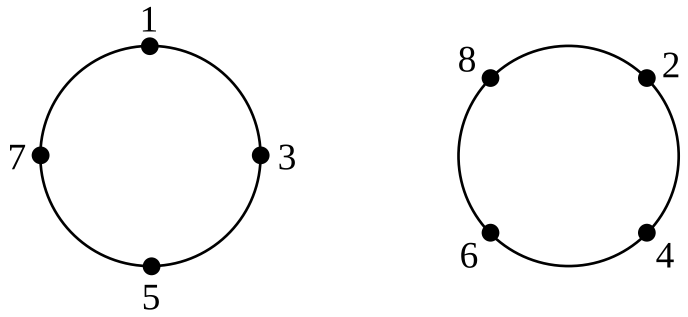
【Suggested Solution 2】

Refer to the diagram below, since we can select 0, 1 or 2 numbers in a group, then the total number of ways to select per group is $1 + 4 + 2 = 7$ ways. (5 points)

So the total number of ways to select numbers is $7 \times 7 = 49$ ways. (5 points)

Then we subtract the cases wherein we select 0 at both groups, as well as taking only 1 total number from both groups.(5 points)

Therefore, we have a total of $49 - 4 - 4 - 1 = 40$ ways. (5 points)



Answer：40 ways