Solution to
Sixth International Mathematics Assessment for Schools
Round 1 of Junior Division

1. What is the simplified value of $\sqrt{(-18)^2}-1^{2016}-(-1)^{2017}$?

(A) –20 (B) –18 (C) 0 (D) 16 (E) 18

【Solution】

$\sqrt{(-18)^2}-1^{2016}-(-1)^{2017}=\sqrt{18^2}-1-(-1)=18-1+1=18$.

Answer: (E)

2. Which of the following numbers is the sum of four consecutive positive integers?

(A) 2016 (B) 2017 (C) 2018 (D) 2019 (E) 2020

【Solution 1】

Let the smallest of the four consecutive positive integers be a, then the sum of the four consecutive positive integers is

$$a+(a+1)+(a+2)+(a+3)=4a+6=4\times(a+1)+2.$$

Since the remainder of the sum of the four consecutive integers when divided by 4 is always 2. Among the options, only 2018 satisfy this condition.

【Solution 2】

The sum of four consecutive positive integers must be even number, hence (B) and (D) are impossible. The mean of four consecutive positive integers must not a integers, hence (A) and (E) are impossible. However, $2018\div 4=504.5$, those four consecutive positive integers be 503, 504, 505, 506. So, (C) satisfy this condition.

Answer: (C)

3. In a supermarket, 3 kg of pear costs \$16.26, while 2 kg of apple costs \$13.62. How much does 1 kg of apple cost more than 1 kg of pear?

(A) 0.61 (B) 1.39 (C) 1.42 (D) 1.81 (E) 2.64

【Solution】

Since the cost of pear per kg is $\$16.26\div 3=5.42$, and the cost of apple per kg is $\$13.62\div 2=6.81$, then the difference is $\$6.81-\$5.42=\$1.39$.

Answer: (B)

4. The value of the faction $\frac{m}{n}$ increases by 1 when the numerator increases by 2017. Find the value of n.

(A) 1 (B) 2016 (C) 2017 (D) 2018 (E) Uncertain

【Solution】

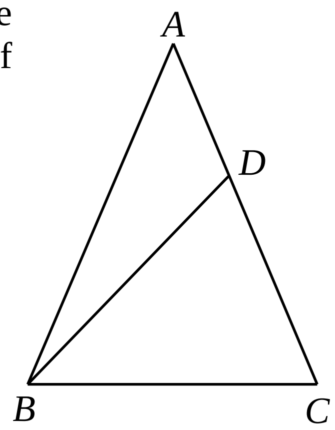
From the given information, we know that $\frac{m+2017}{n}=\frac{m}{n}+1$, so $\frac{2017}{n}=1$ and hence $n=2017$.

Answer: (C)

5. In the figure below, triangle ABC is an isosceles triangle with $AB = AC$. Point D is on AC with $BD = BC$. If $\angle ABD = 21°$, what is the measure, in degrees, of $\angle BAC$?

(A) 21 (B) 38 (C) 42

(D) 46 (E) 54



【Solution 1】

From the given information, $AB = AC$, it follows $\angle C = \angle ABC = \angle DBC + 21°$, $\angle DBC = \angle C - 21°$. Since $BD = BC$, then $\angle C = \angle BDC$, this implies $\angle DBC = 180° - 2\angle C$. Thus $180° - 2\angle C = \angle C - 21°$, so $\angle C = 67°$, finally we have $\angle BAC = 180° - 2\angle C = 46°$.

【Solution 2】

Since the exterior angle of a triangle is equal to sum two non-adjacent interior angles, it follows $\angle BDC = \angle BAC + \angle ABD = \angle BAC + 21°$. By $AB = AC$, we have $\angle C = \angle ABC = \angle DBC + 21°$, that is $\angle BAC = \angle DBC$. This implies $180° = 2\angle ACB + \angle BAC = 2(\angle BAC + 21°) + \angle BAC = 3\angle BAC + 42°$, $\angle DBC = 46°$.

Answer: (D)

6. What is the sum of all the prime divisors in the final result of $2^3 + 0^3 + 1^3 + 7^3$ (repeated prime divisors are counted only once)?

(A) 7 (B) 12 (C) 13 (D) 16 (E) 64

【Solution】

$2^3 + 0^3 + 1^3 + 7^3 = 8 + 0 + 1 + 343 = 352 = 2\times2\times2\times2\times2\times11$, thus $2 + 11 = 13$.

Answer: (C)

7. Let a, b, c, d, e and f are distinct digits such that the expression $\overline{ab} + \overline{cd} = \overline{ef}$.

What is the least possible value of $\overline{ef}$?

(A) 30 (B) 34 (C) 36 (D) 39 (E) 41

【Solution】

Since $a + c \geq 1 + 2 = 3$, one has $e \geq 3$. Take $e = 3$, one gets a and c are 1 and 2, respectively. Moreover b and d are non-zero, otherwise f is equal to b or d. So $b + d \geq 4 + 5 = 9$, that is, $f = 9$. For example, $14 + 25 = 39$ and $15 + 24 = 39$.

Answer: (D)

8. How many integers x satisfy $|2x+1| \leq 8$?

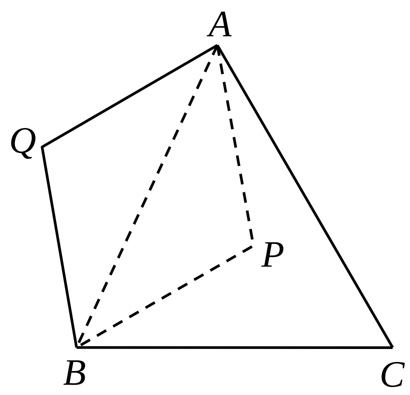
(A) 3 (B) 4 (C) 6 (D) 8 (E) 9

【Solution】

According to Principle of Absolute Inequality, we have $-8 \leq 2x + 1 \leq 8$, it follows $-9 \leq 2x \leq 7$, or $-4.5 \leq x \leq 3.5$, this implies x takes value from -4, -3, -2, -1, 0, 1, 2, 3; hence there are 8 integers satisfy the given inequalities.

Answer: (D)

9. In the figure below, $AQBC$ is a convex quadrilateral with $QA=QB$ and $\angle C=60°$. Triangle QAB is folded along AB to form triangle PAB. It is also known that $\angle PBC=30°$ and $\angle PAC=20°$. What is the measure, in degrees, of $\angle AQB$?

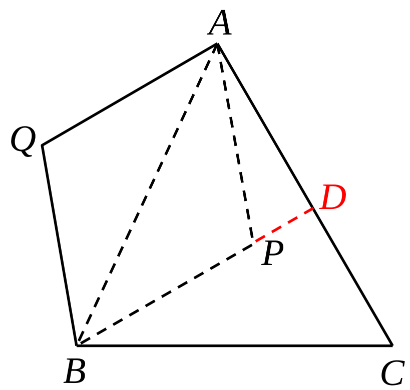


(A) 100 (B) 110 (C) 120
(D) 130 (E) 140

【Solution 1】

Since three interior angles of ΔABC equals $180°$, then $\angle PAB+\angle PBA=180°-(60°+20°+30°)=70°$, so $\angle APB=180°-70°=110°$. As $\Delta PAB\cong\Delta QAB$ by folding along AB, hence $\angle AQB=110°$.

【Solution 2】



Extend BP meet AC at point D.
Since $\angle BDC=180°-30°-60°=90°$,
we can get $\angle APD=180°-90°-20°=70°$,
hence $\angle AQB=\angle APB=180°-70°=110°$.

Answer: (B)

10. If all the edges of a rectangle are integers, then which of the following **CANNOT** be the length of its diagonals?

(A) 5 (B) 6 (C) $\sqrt{41}$ (D) $\sqrt{53}$ (E) 10

【Solution】

By Pythagoras's theorem and the given information, square length of the diagonal is equal to sum of squares of two positive integers. Since $5^2=3^2+4^2$, $\sqrt{41}^2=4^2+5^2$, $\sqrt{53}^2=2^2+7^2$, $10^2=6^2+8^2$, while among five numbers $1^2=1$, $2^2=4$, $3^2=9$, $4^2=16$ and $5^2=25$, no two sums yield $6^2=36$, so 6^2 is not the sum of squares of two positive integers.

Answer: (B)

11. The length of 2 altitudes on adjacent sides of a parallelogram are 2 cm and 3 cm respectively with perimeter of the parallelogram is 18 cm. What is the area, in cm^2, of the parallelogram?

(A) 9.6 (B) 10 (C) 10.5 (D) 10.8 (E) 12

【Solution 1】

Suppose lengths of two adjacent sides of the given parallelogram as x cm, y cm and area is S, then $S=2x=3y$, it follows $x=\frac{S}{2}$, $y=\frac{S}{3}$. Since the perimeter of the parallelogram is 18 cm, so that $2\left(\frac{S}{2}+\frac{S}{3}\right)=18$, then $S=10.8\,\text{cm}^2$.

【Solution 2】

Suppose lengths of two sides of the given parallelogram are x cm, y cm, then $2x=3y$, so that $x:y=3:2$. But the perimeter is given as 18 cm, this implies $x=\frac{18}{2}\times\frac{3}{5}=\frac{27}{5}$, thus the area is $\frac{27}{5}\times 2=\frac{54}{5}=10.8\,\text{cm}^2$.

Answer: （D）

12. Three students are to participate in four games. Each student participates in at least one game and each game has exactly one student participating. In how many ways can this be done?

（A）12　　（B）18　　（C）24　　（D）30　　（E）36

【Solution 1】

From the given information, one student will participate in two events and each of other two students participate on exactly one event. Let the student (the one will participate in 2 events) choose the events that he wants to register, then there are 4 ways of registering the two events, follow by the first student with one event has 3 ways to register his event, the second student with one event has also 3 ways to choose his event. Thus, the total number is $3\times4\times3=36$ ways of participating.

【Solution 2】

From the given information, one student attends 2 events and two others student each attends one. There are three ways to choose the student attending two events. The student with two events has $\frac{4\times3}{2}=6$ ways to choose his events. The other two student, choosing from remaining two events, has totally 2 ways to choose their students. Hence, the total number is $3\times6\times2=36$.

Answer: （E）

13. The sum of a, b and c is negative, while the product of these three numbers is positive. If $x=\frac{a}{|a|}+\frac{b}{|b|}+\frac{c}{|c|}$, what is the value of $x^{2017}-2017x^2+36$?

（A）-1982　　（B）-1981　　（C）-1980　　（D）1980　　（E）1982

【Solution】

Since the product of given three numbers a, b, c are positive, we know this will happen when all three are positive or only one is positive, while the other two are negative. For the sum of all positive numbers cannot be negative, two of a, b, c are negative. Thus $x=1+(-1)+(-1)=-1$, and $x^{2017}-2017x^2+36=-1-2017+36=-1982$.

Answer: （A）

14. Given two positive integers, $\frac{4}{7}$ of the first is exactly $\frac{2}{5}$ of the second. What is the minimum sum of these two integers?

（A）10　　（B）14　　（C）15　　（D）35　　（E）17

【Solution】

Write the two integers as x and y, with $\frac{4}{7}x=\frac{2}{5}y$, one has $\frac{x}{y}=\frac{7}{10}$. Since x and y are both positive integers, their sum is at least $7+10=17$.

Answer: (E)

15. Tom starts working at 9∶00 in the morning and finishes at 5∶00 in the afternoon. How many more degrees does the minute hand rotates than the hour hand does on the clock during this period?

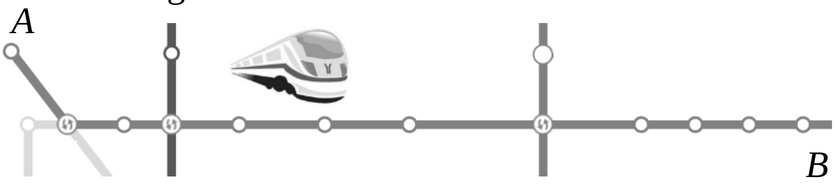
(A) 120 (B) 1200 (C) 1320 (D) 2640 (E) 2880

【Solution】

Tom works for 8 hours. In this period, the minute hand rotates 8 rounds, that is $8\times 360=2880$ degrees. And the hour hand rotates $\frac{8}{12}=\frac{2}{3}$ round, that is $\frac{2}{3}\times 360=240$ degrees. The difference is $2880-240=2640$ degrees.

Answer: (D)

16. The price criteria of the subway ticket of a city is as follows: \$2 for within 4 km, \$1 more per 4 km for distances between 4 km and 12 km, \$1 more per 6 km for distances over 12 km. It costs \$8 to take subway from station A to station B. Which of the following is closest to the distance between A and B?



(A) 12 km (B) 18 km (C) 24 km (D) 36 km (E) 48 km

【Solution】

The cost of a 12km trip is $2+\frac{(12-4)}{4}=4$ dollars, so the distance between A and B is over 12 km, and the cost increases by \$4 after 12 km. So the distance after 12 km is at least $6\times 4=24$ km but less than $6\times 5=30$ km. Then the distance between station A and station B is at least 36 km, and at most 42 km. Among the options, the closest distance between A and B is 36 km.

Answer: (D)

17. Consider a 3-digit number $\overline{abc}$, where a, b and c are distinct digits, so that their sum is 7. How many such three-digit numbers are there?

(A) 6 (B) 12 (C) 14 (D) 18 (E) 22

【Solution】

From the given information, $\overline{abc}$ is a three-digit number, it follows $a\neq 0$. When one of a, b, c is 0, there are three combination: 0, 1, 6; 0, 2, 5 or 0, 3, 4, each combines to 4 possible three-digit numbers. When all three are non-zero, they must be 1, 2, 4, which combines to 6 three-digit numbers. Totally we have $3\times 4+6=18$ distinct three-digit numbers.

Answer: (D)

18. If positive numbers x and y satisfy $x^2 - y^2 = 2$. What is the simplified value of $x\sqrt{2+y^2} - y\sqrt{x^2-2}$?

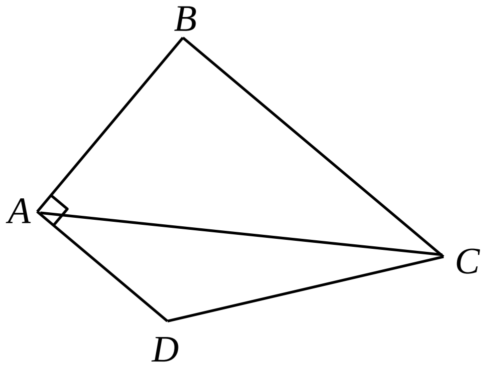
(A) 2 (B) $2\sqrt{2}$ (C) 4 (D) $4\sqrt{2}$ (E) Uncertain

【Solution】

$x\sqrt{2+y^2} - y\sqrt{x^2-2} = x\sqrt{x^2} - y\sqrt{y^2} = x^2 - y^2 = 2$.

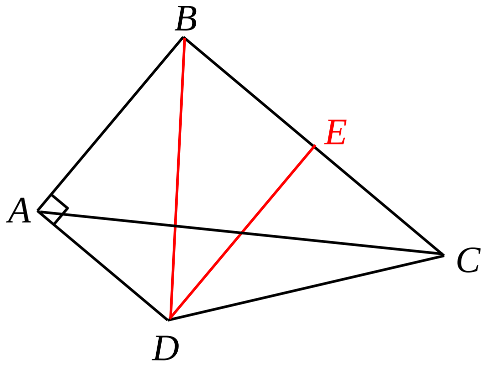
Answer: (A)

19. In the figure below, $ABCD$ is a quadrilateral where $AB = 4$ cm, $BC = 6$ cm, $CD = 5$ cm, $DA = 3$ cm and $\angle BAD = 90°$. Find the length, in cm, of AC.



(A) 5 (B) 7 (C) $\sqrt{34}$ (D) $3\sqrt{5}$ (E) $2\sqrt{13}$

【Solution】



Connect BD, then $BD = \sqrt{3^2 + 4^2} = 5 = DC$.

Let the midpoint of BC be E, then $DE \perp BE$, and $BE = \frac{1}{2}BC = 3 = AD$, so $\triangle BED \cong \triangle DAB$, and hence $\angle EBD = \angle ADB$, then $BC // AD$, and $BC \perp AB$. By Pythagoras's Theorem

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{4^2 + 6^2} = 2\sqrt{13}.$$

Answer: (E)

20. If $a = \frac{1}{3}$ and $b = \frac{1}{4}$, find the numerical value of $a^3 + b^3 - a^2b - ab^2$.

(A) $\frac{7}{1728}$ (B) $\frac{7}{1718}$ (C) $\frac{5}{1718}$ (D) $\frac{5}{1728}$ (E) $\frac{5}{1628}$

【Solution】

$a^3 + b^3 - a^2b - ab^2 = (a+b)(a-b)^2 = (\frac{1}{3} + \frac{1}{4}) \times (\frac{1}{3} - \frac{1}{4})^2 = \frac{7}{12} \times (\frac{1}{12})^2 = \frac{7}{1728}$.

Answer: (A)

21. Let *a*, *b* and *c* be positive integers satisfy $a^2+bc=\frac{19}{a}+b+c$. Find the sum of *a*, *b* and *c*.

【Solution】

From the given information, $\frac{19}{a}$ is an integer, so $a=1$ or $a=19$.

When $a=1$: $bc+1=b+c+19$, and $(b-1)(c-1)=19$. Since 19 is a prime, one of $b-1$, $c-1$ is 1 and the other is 19, so $a+b+c=1+(1+1)+(19+1)=23$.

When $a=19$: $bc+361=b+c+1$, and $(b-1)(c-1)=-359$, this implies one of $b-1$, $c-1$ is negative, which is a contradiction.

Answer: 023

22. The operation「$\otimes$」satisfies:

(i) For all *x* and *y*, $x\otimes y=(x-1)\otimes(y-1)+x+y$;

(ii) For all *x*, $x\otimes 1=1$.

Find the value of $(3\otimes 3)\otimes 3$.

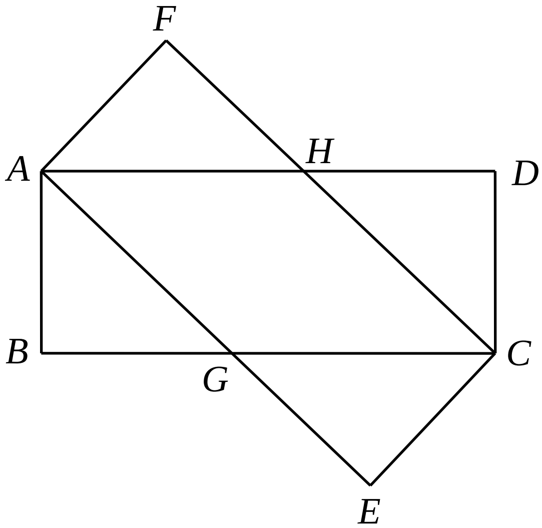
【Solution】

$3\otimes 3=2\otimes 2+3+3=1\otimes 1+2+2+3+3=11$,

so $(3\otimes 3)\otimes 3=11\otimes 3=10\otimes 2+11+3=9\otimes 1+10+2+11+3=27$.

Answer: 027

23. In the figure below, *ABCD* and *AFCE* are congruent rectangles with $AB=AF=20$ cm and $AD=AE=50$ cm. Find the area, in cm^2, of *AGCH*.

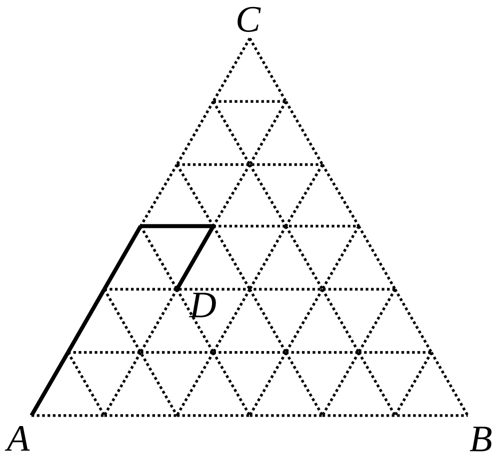


【Solution】

By Property of Symmetry, *AGCH* is a rhombus, let us denote its length as *x* cm, then $BG=50-CG=50-x$, $AG=x$. In the triangle *ABG*, the Pythagoras' theorem gives $20^2+(50-x)^2=x^2$, which solves to $x=29$. So the area of *AGCH* is $29\times 20=580$ cm^2.

Answer: 580

24. In the figure below, the side length of equilateral triangle ABC is 6 cm. Each side is divided into 6 equal segments and connects corresponding dividing points to get an equilateral network. Call a point "reachable" if it can be connected to A by a broken line of length 5 cm along the grid lines without passing any lattice point twice. For example, point D in the figure is reachable. Find the number of reachable points in the figure.



【Solution】

Let us name those connecting to A by a grid line segment of length 1 cm as "one step". From the figure, we know it needs at least 6 steps to reach a point on line BC. Then points on BC are not reachable. Actually, it is easy to discover that any other point is reachable. Some paths are as follows:

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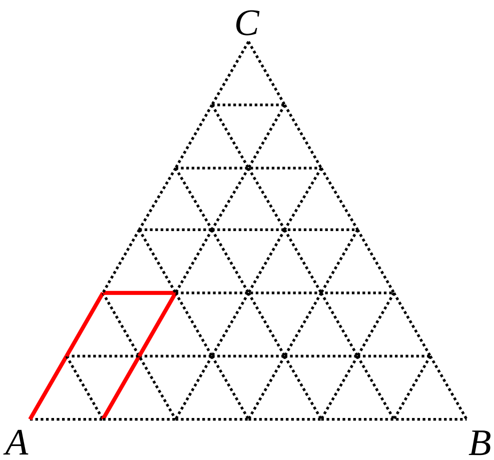
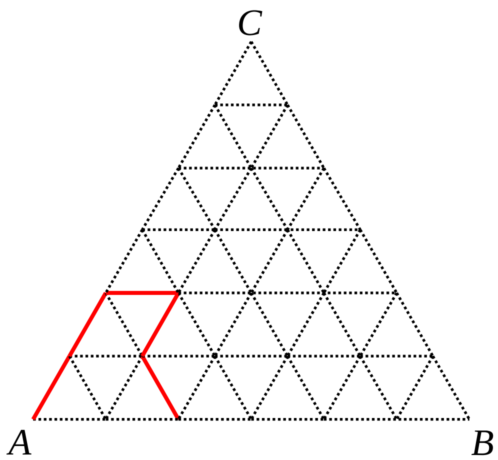
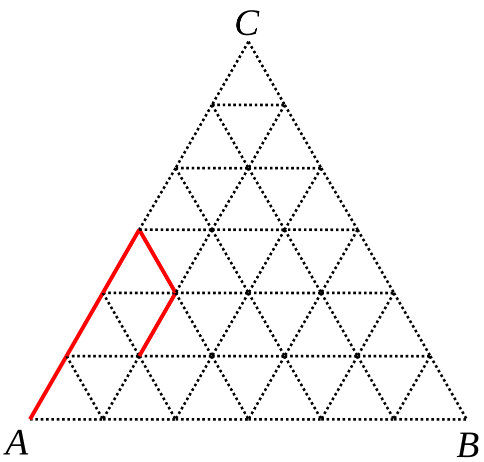
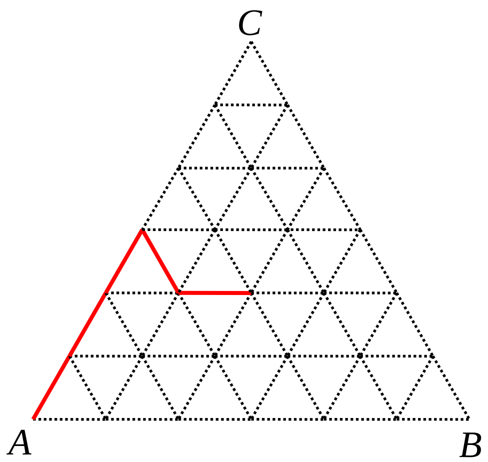








So there are a total of $2+3+4+5+6=20$ reachable lattice points.

Answer: 020

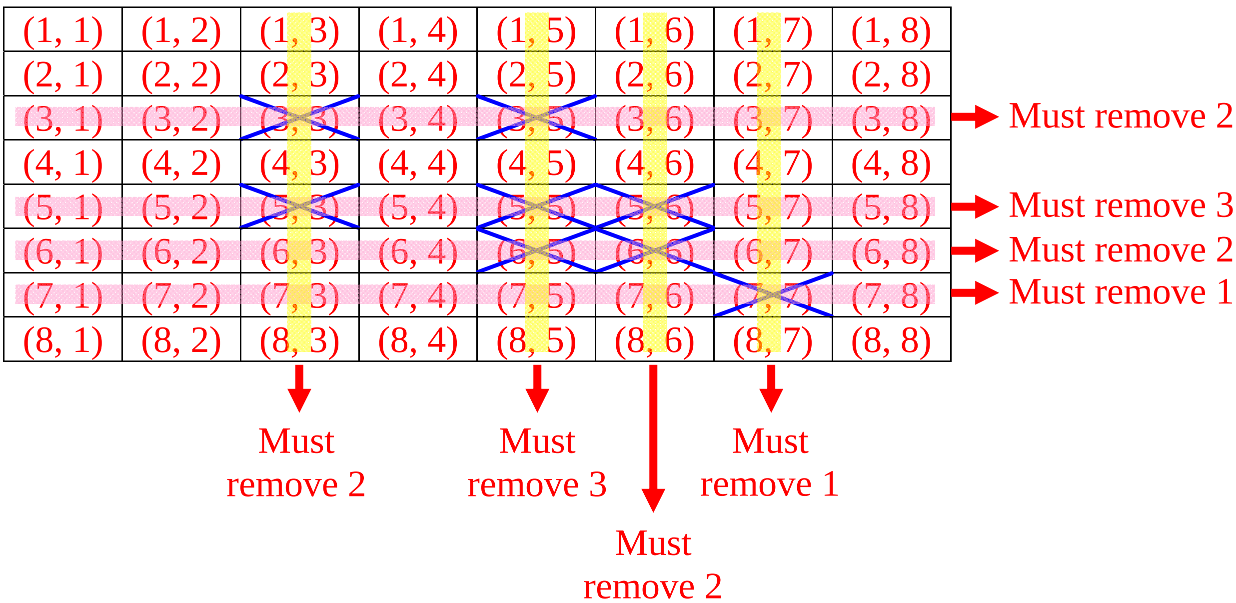
25. The students in a research class are clustered into two groups: the morning and afternoon sessions. A student takes part in exactly one group in each session (the two groups in each session can be different and the number of students in each group can be different). Each group has at least one student and at most 8 students. Each student reports the number of students in the group he or she belongs to in two sessions. One finds that no two students report the same pair of numbers (with order, for example, (1, 4) and (4, 1) are different). What is maximum number of students in the class?

【Solution】

Consider the two numbers reported by a student as an ordered pair. The first number represents number of students in the morning session, the second number represents the afternoon session. Since maximum number of students in a group is 8, there is a total of 64 combinations as follows:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, 7)	(1, 8)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, 7)	(2, 8)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)	(3, 7)	(3, 8)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)	(4, 7)	(4, 8)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)	(5, 7)	(5, 8)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)	(6, 7)	(6, 8)
(7, 1)	(7, 2)	(7, 3)	(7, 4)	(7, 5)	(7, 6)	(7, 7)	(7, 8)
(8, 1)	(8, 2)	(8, 3)	(8, 4)	(8, 5)	(8, 6)	(8, 7)	(8, 8)

Since the pairs in row k $(1 \le k \le 8)$ corresponds students in a group of size k in the morning, the total number of students in row k is a multiple of k. There are at most 6 students in row 3, 5 students in row 5, 7 students in row 7. Similarly, the number of students in column k is a multiple of k. If (3, 3), (3, 5), (5, 3), (5, 5), (5, 6), (6, 5), (6, 6), (7, 7) are taken out as without students, the remaining 56 pairs satisfy the above requirement. Group of the students in two sessions row-wise and column-wise respectively, one gets a feasible grouping. So the maximum number of students is 56.



Answer: 056