Solution Key to Second Round of IMAS 2017/2018
Middle Primary Division

1. Arranged 30 triangles in a row and color them black and white in a pattern as shown below. How many more black triangles than white triangles are there?

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（A）8 （B）10 （C）12 （D）14 （E）15

【Solution】

From the picture, it shows that the coloring is periodic with period 3, consisting of 2 black triangles and 1 white triangle. In each period, the number of black triangles is one more than the white triangle. Since there are $30 \div 3 = 10$ periods, so there are 10 more black triangles.

Answer: （B）

2. It is known that $A \times B \times C = 30$, $B \times C \times D = 90$ and $C \times D \times E = 120$, what is the value of $A \times C \times E$?

（A）20 （B）30 （C）40 （D）50 （E）60

【Solution】

$A \times C \times E = (A \times B \times C) \times (C \times D \times E) \div (B \times C \times D)$, so $A \times C \times E = 30 \times 120 \div 90 = 40$.

Answer: （C）

3. Replace each Δ by "＋" or "－" in the expression $1\Delta 2\Delta 3\Delta 8\Delta 15$ and compute the value of the expression. How many different positive integers can we have for the expression?

（A）8 （B）10 （C）12 （D）14 （E）16

【Solution 1】

Since $1+2+3+8=14<15$, so if the sign placed before 15 is negative, the result is not positive. When the sign before 15 is a plus sign and since 1 is always positive, the result is always positive. The difference between 2,3 and 8 are all distinct, so the result for different sign choices are distinct. Thus, there are $2 \times 2 \times 2 = 8$ different positive integers.

【Solution 2】

There are $2 \times 2 \times 2 \times 2 = 16$ ways in filling the signs.

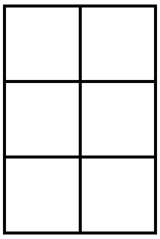
(1) $1+2+3+8+15=29$;
(2) $1+2+3+8-15=14-15<0$, it is non-positive;
(3) $1+2+3-8+15=13$;
(4) $1+2-3+8+15=23$;
(5) $1-2+3+8+15=25$;
(6) $1+2+3-8-15=6-23<0$, it is non-positive;
(7) $1+2-3+8-15=11-18<0$, it is non-positive;
(8) $1-2+3+8-15=10-15<0$, it is non-positive;
(9) $1+2-3-8+15=7$;
(10) $1-2+3-8+15=9$;

(11) $1-2-3+8+15=19$;
(12) $1+2-3-8-15=3-26<0$, it is non-positive;
(13) $1-2+3-8-15=4-25<0$, it is non-positive;
(14) $1-2-3+8-15=9-20<0$, it is non-positive;
(15) $1-2-3-8+15=3$;
(16) $1-2-3-8-15=1-28<0$, it is non-positive;

Therefore, we have 29, 13, 23, 25, 7, 9, 19 and 3, a total of 8 different positive integers.

Answer: (A)

4. A rectangle is divided into 6 identical squares as shown in the figure below. Color four squares black such that each row has at least one black square. Two color methods are considered the same if they are identical after rotation. How many different color methods are there?

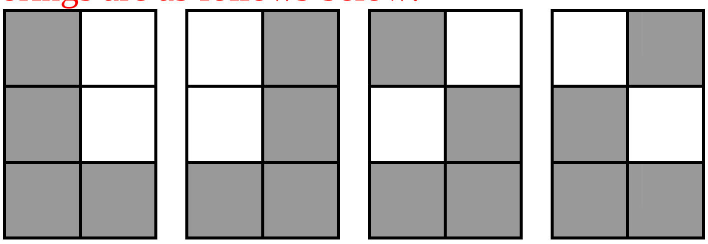


(A) 4 (B) 5 (C) 6 (D) 7 (E) 10

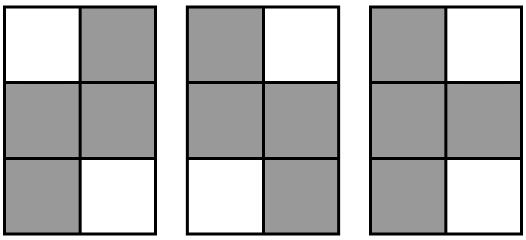
【Solution】

There are three rows in the figure and in every scenario, two of the rows will have one black square and the third row will have two black squares.

(i) If the number of black squares are 1, 1 and 2 respectively in rows from top to bottom, the colorings are as follows below:



(ii) If the number of black squares are 1, 2, and 1 respectively from top to bottom, the colorings are as follows below:



(iii) If the number of black squares are 2, 1 and 1 respectively, the coloring is rotationally equivalent to cases in (i).

In total, there are 7 coloring methods.

Answer: (D)

5. Two cars start moving from opposite ends of a road towards each other at a constant speed. The faster car travels at 40 km per hour. After two hours, the faster car is 20 km past the midpoint and 6 km away before meeting the slower car. What is the speed of the slower car in km per hour?

（A）17 （B）19 （C）21 （D）23 （E）25

【Solution 1】

The faster car covers half the length of the road plus 20 km in two hours, while the slower car covers 26 km less than the half length of the road. Thus, the faster car covers 46 km more than the slower car in two hours. The speed difference is $46 \div 2 = 23$ km and the speed of the slower car is $40 - 23 = 17$ km.

【Solution 2】

The faster car travels $40 \times 2 = 80$ km in two hours, which is 20 km more than half length of the road. Then, the road has a length of $2 \times (80 - 20) = 120$ km. The distance of the two cars is 6 km, so the slower car travels $120 - 80 - 6 = 34$ km for two hours. Its speed is then $34 \div 2 = 17$ km.

Answer: (A)

6. Select three different numbers from 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12 such that their average is 5. How many different possible combinations are there?

【Solution】

Write the sum of three numbers, which is $5 \times 3 = 15$, as the sum of three distinct positive integers. Enumerating all possible ways, we get,

$$15 = 12 + 2 + 1 = 11 + 3 + 1 = 10 + 4 + 1 = 10 + 3 + 2 = 9 + 5 + 1 = 9 + 4 + 2$$
$$= 8 + 6 + 1 = 8 + 5 + 2 = 8 + 4 + 3 = 7 + 6 + 2 = 7 + 5 + 3 = 6 + 5 + 4$$

In total, there are 12 different ways.

Answer: 12

7. The pages of a book begin from No. 1, 2, 3, ⋯ such that two consecutive numbers appear on both sides of a single page. When one page is torn, the sum of the remaining page numbers is 1133. What is the sum of those two numbers on the torn page?

【Solution 1】

If the book is until page 50, the sum of all page numbers is $1 + 2 + 3 + \cdots + 49 + 50 = 1275$, but $1275 - 1133 = 142$ and there is no page before 50 has page numbers that has a sum of 142. This will result that the torn page is higher than 50, so there is no possible solution;

If the book is until page 49, the sum is $1 + 2 + 3 + \cdots + 48 + 49 = 1225$, and $1225 - 1133 = 92$ is even, this is not possible for a sum of page numbers of a single page;

If the book is until page 48, the sum is $1 + 2 + 3 + \cdots + 47 + 48 = 1176$, and $1176 - 1133 = 43$, $43 = 21 + 22$, the torn page has numbers 21 and 22;

If the book in until less than or equal to page 47, sum of all page number is at most $1 + 2 + 3 + \cdots + 46 + 47 = 1128 < 1133$, which is not possible.

Thus, the sum of page numbers of the torn page is $43 = 21 + 22$.

【Solution 2】

The two numbers on one page is consecutive, whose sum is odd. The total sum of all page numbers is the sum of 1133 and an odd number, which is even. The sum of all page numbers is sum of consecutive integers starting from 1, the total page numbers is a multiple of 4 or a multiple of 4 plus 3. $1+2+3+\cdots+46+47=1128<1133$, then, the book has at least 48 pages. Possible pages numbers are 48, 51, 52, 55, 56, $\cdots$.

When the book is at least 51 pages, tearing up one page leaves at least 49 different page numbers, whose sum is at least $1+2+3+\cdots+48+49=1225>1133$, which is not possible for a solution.

For a book with 48 pages, $1+2+3+\cdots+47+48=1176$, but $1176-1133=43=21+22$, this satisfies the condition of the problem.

Thus, the sum of page numbers of the torn page is $43=21+22$.

Answer: 43

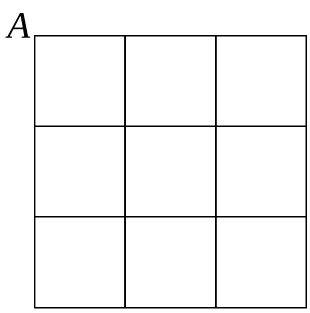
8. The number of boys in a class is twice as the number of girls. In a math test, the average score of the class is 86, but average score of the girls is 90. What is the average score of the boys?

【Solution】

Divide all students into several groups so that each group has one girl and two boys. Averagely, the sum of the scores of the three students in a group is $86\times3=258$ and the score of the girl is 90. Then the total score of the two boys is $258-90=168$, whose average is $168\div2=84$.

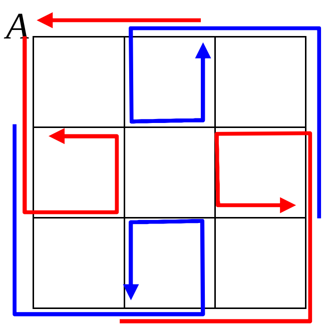
Answer: 84

9. The road map of a certain city consists of 3×3 squares with each of the smaller squares having a side length of 20 meters as shown in the figure. A street sweeper starts cleaning from point *A* and every road thereafter until finally returning to *A*. What is the shortest distance, in meters, in which he can do the task?



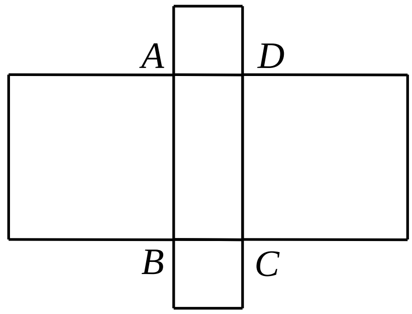
【Solution】

Eight intersections of the road map are the intersection points of odd number of roads. To walk through each road at least once, 4 paths connecting pairs of the odd intersections must be walked through twice. So, the sweeper must walk at least $(24+4)\times20=560$ m. One cleaning path is shown below.



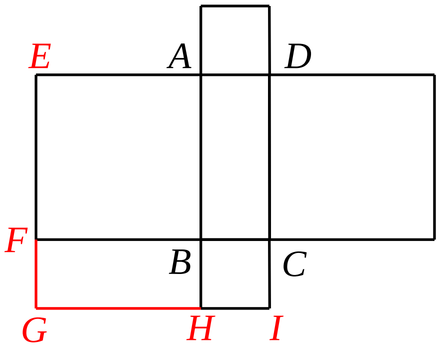
Answer: 560

10. The perimeter of rectangle *ABCD* is 34 cm. Draw four squares extending outward from its four sides, as shown in the figure below. If the sum of the areas of the four new squares is 338 cm^2, what is the area, in cm^2, of rectangle *ABCD*?



【Solution】

Mark each point as in the figure below and draw the rectangle *BFGH*. The rectangles *BFGH* and *ABCD* are congruent. Quadrilateral *EGID* has four inner right angles and both side lengths equal to sum of two sides of *ABCD*, so it is a square with side length $34 \div 2 = 17$ cm and area $17 \times 17 = 289$ cm^2. The sum of areas of *EFBA* and *BHIC* is $338 \div 2 = 169$ cm^2, so the area of *ABCD* is $(289 - 169) \div 2 = 60$ cm^2.



Answer: 60 cm^2

11. Millie has 60 red beads, 50 black beads and a magic machine. Each time 4 red beads are inserted into the machine, it gives out 1 black bead and for every 5 black beads is inserted, the machine gives out 2 red beads. After having operated the machine 30 times, Millie has no more red beads. Find the number of black beads she has now.

【Solution】

Operation (i): 4 red beads are inserted into the machine, it gives out 1 black bead.
Operation (ii): 5 black beads are inserted into the machine, it gives out 2 red beads.
Each operation reduces the number of total beads by 3. Since Millie has $50 + 60 = 110$ beads at the start, after 30 operations he has $110 - 30 \times 3 = 20$ beads left, all of which are black. One operation sequence could be to operate step (ii) for 10 times to get 80 red beads and 0 black beads; then to operate step (i) for 20 times to get 20 black beads and 0 red beads.

Answer: 20

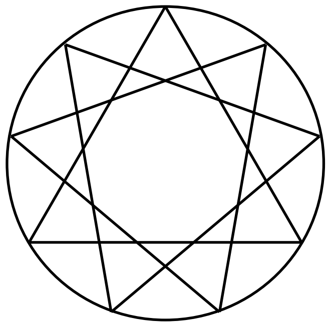
12. A 3-digit number $\overline{abc}$ is said to be a lucky number if $a \times b \times c$ is also a three-digit number. What is the least possible lucky number $\overline{abc}$?

【Solution】

First, we have that both b and c are no more than 9, thus $b\times c\leq 81$. In order that $a\times b\times c\geq 100$, one has $a\geq 2$. When $a=2$, in order that $a\times b\times c\geq 100$, $9\times b$ is at least 50, then $b\geq 6$. When $a=2$, $b=6$, $a\times b=12$, in order that $a\times b\times c\geq 100$, $c\geq 9$. The least possible $\overline{abc}$ is 269, while $a\times b\times c=108$.

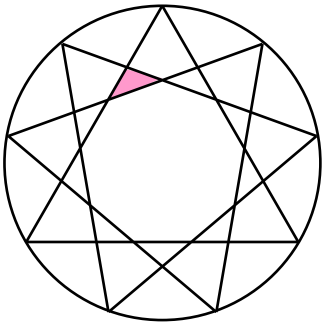
Answer: 269

13. Rotate an equilateral triangle inscribed in a circle 40 degrees clockwise and counter-clockwise, as in the figure below. How many triangles are there in the figure?

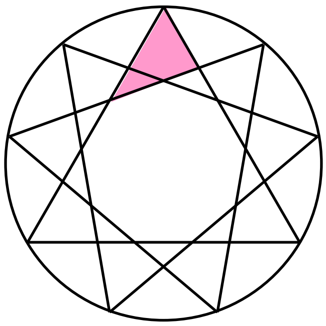


【Solution】

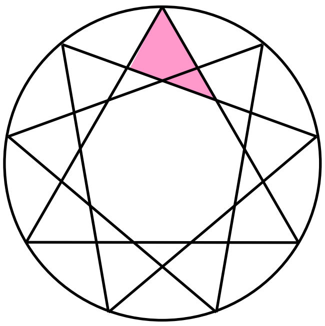
(i) There are 9 triangles of same size but in different positions as the shaded triangle in the figure below.



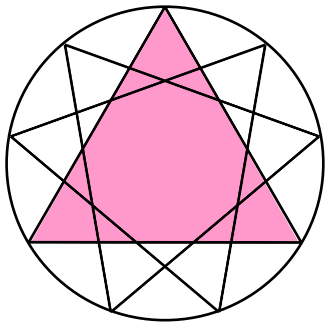
(ii) There are 9 triangles of same size but in different positions as the shaded triangle in the figure below.



(iii) There are 9 triangles of same size but in different positions as the shaded triangle in the figure below.



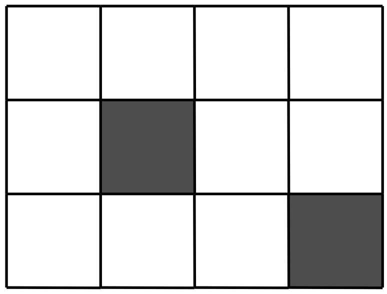
(iv) There are 3 triangles of same size but in different positions as the shaded triangle in the figure below.



Totally there are $9+9+9+3=30$ triangles.

Answer: 30.

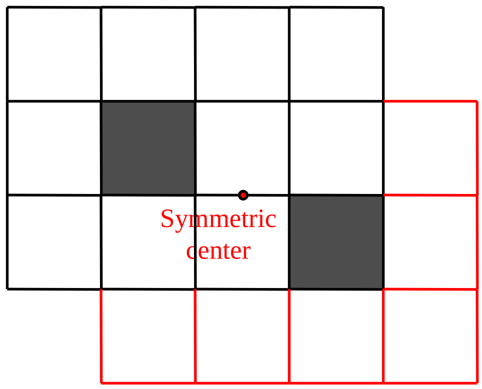
14. A rectangle is divided into 12 unit squares such that 10 are white and 2 are black, as shown in the figure below. To form a centrally symmetric picture by adding some white squares but no black squares, what is the least number of white squares needed? Please draw the centrally symmetric picture.



【Solution】

Since no more black unit squares are added, the symmetric center of the whole picture is the symmetric center of the two black unit squares. (5 points)

4 white unit squares are already symmetric to one another with respect to this center. 6 more white unit squares are needed to be symmetric to those 6 alone white unit squares, (5 points) as in the figure to the right. (10 points)



Answer: 6

15. There are 12 different positive integers which satisfy the condition that the product of every 5 numbers is even and that the sum of all 12 numbers is odd. Find the least possible sum of these 12 positive integers.

【Solution】

Since every 5 numbers, the product should be even, at most 4 of the 12 numbers is odd. (5 points)But since the sum of all the numbers is odd, there is an odd number of odd numbers among them. So, the number of odd numbers is 1 or 3. (5 points)

(i) If there is only 1 odd number, taking sum of the smallest odd number and 11 smallest even numbers, one gets

$$1+2+4+6+8+10+12+14+16+18+20+22=133.\text{(5 points)}$$

(ii) If there are 3 odd numbers, taking sum of the smallest 3 odd number and 9 smallest even numbers, one gets

$$1+3+5+2+4+6+8+10+12+14+16+18=99.$$

So, the least possible sum of these 12 positive integers is 99. (5 points)

Answer: 99