Solution to
Seventh International Mathematics Assessment for Schools
Round 1 of Junior Division

1. Calculate the value of the expression: $2020^2-2019^2-\sqrt{(-2018)^2}$.

(A) 2021 (B) 2022 (C) 2037 (D) 4039 (E) 6057

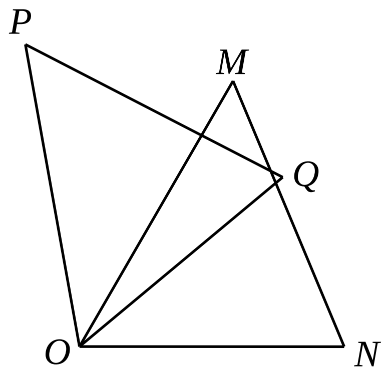
【Solution】

$$\begin{aligned}2020^2-2019^2-\sqrt{(-2018)^2}&=(2020+2019)(2020-2019)-2018\\&=2020+2019-2018\\&=2021\end{aligned}$$

Hence (A).

Answer：(A)

2. In the figure below, it is known that $\triangle POQ\cong\triangle MON$, $\angle PON=100°$ and $\angle MOQ=20°$. What is the measure, in degrees, of $\angle POQ$?



(A) 20 (B) 30 (C) 40 (D) 45 (E) 60

【Solution】

Since $\angle POQ=\angle MON$, and $\angle POQ+\angle MON=\angle PON+\angle MOQ=120°$, $\angle POQ=60°$. Hence (E).

Answer：(E)

3. If $x=2$ and $y=3$, then what is the value of $x^4+y^4-x^3-y^3+x^2+y^2$?

(A) 71 (B) 72 (C) 75 (D) 83 (E) 85

【Solution】

$$\begin{aligned}x^4+y^4-x^3-y^3+x^2+y^2&=2^4+3^4-2^3-3^3+2^2+3^2\\&=16+81-8-27+4+9\\&=75\end{aligned}$$

Hence (C).

Answer：(C)

4. Two positive integers m and n satisfy the following conditions: When m is divided by 35, it leaves a remainder of 12 and when n is divided by 21, it leaves a remainder of 15. What is the remainder when $m-n$ is divided by 7?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

【Solution 1】

Write $m=35a+12$, $n=21b+15$, then $m-n=35a-21b-3=7(5a-3b-1)+4$, that is, $m-n$ has remainder 4 divided by 7. Hence (C).

【Solution 2】

m has remainder 12 divided by 35, so m has remainder 5 divided by 7; similarly, n has remainder 15 divided by 21, it has remainder 1 divided by 7. The remainder of $m-n$ divided by 7 is $5-1=4$. Hence (C).

Answer：(C)

5. If $x^2-4x+4+\sqrt{xy-2018}=0$, then what is the value of y?

(A) 0 (B) 1009 (C) 2018 (D) 4036 (E) Uncertain

【Solution】

Simplify to $(x-2)^2+\sqrt{xy-2018}=0$, thus $x-2=0$, $xy-2018=0$, i.e. $x=2$ and $y=1009$. Hence (B)

Answer：(B)

6. If $x=3$, then what is the value of $\sqrt{x-1+\sqrt{x-1+\sqrt{x-1+\sqrt{x+1}}}}$?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

【Solution】

When $x=3$,

$$\begin{aligned}\sqrt{x-1+\sqrt{x-1+\sqrt{x-1+\sqrt{x+1}}}} &= \sqrt{3-1+\sqrt{3-1+\sqrt{3-1+\sqrt{3+1}}}}\\ &= \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{4}}}}\\ &= \sqrt{2+\sqrt{2+\sqrt{4}}}\\ &= \sqrt{2+\sqrt{4}}\\ &= \sqrt{4}\\ &= 2\end{aligned}$$

Hence (A).

Answer：(A)

7. The teacher has 2 identical pens and 3 identical pencils to be given out as prizes to two of his students. If each student should receive at least one object, in how many ways can the teacher distribute the prizes?

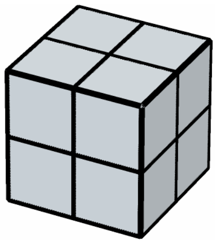
(A) 5 (B) 6 (C) 8 (D) 9 (E) 10

【Solution】

The ways of distributing the pens are (2, 0), (1, 1), (0, 2), and ways of distributing pencils are (3, 0), (2, 1), (1, 2), (0, 3). There are totally 12 ways, where 2 of them has one student got nothing. There are $12-2=10$ ways satisfying the condition of the problem. Hence (E).

Answer：(E)

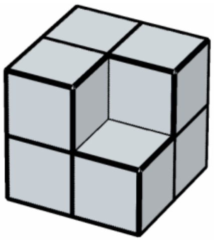
8. As shown in the figure below, a $2\times2\times2$ cube is formed by placing together eight $1\times1\times1$ cubes. If one $1\times1\times1$ cube is removed, what will be the surface area of the remaining figure?



(A) 24 (B) 25 (C) 26 (D) 27 (E) 28

【Solution】

When a small square is taken away, three of its faces are removed, but three other faces become part of new surface, so the surface area of remaining solid is unchanged, which is $6\times2^2=24$ cm^2. Hence (A).



Answer：(A)

9. If positive integers m and n satisfy that $m^2-n^2=13$, then what is the value of m^2+n^2?

(A) 13 (B) 36 (C) 49 (D) 75 (E) 85

【Solution】

$(m+n)(m-n)=13$ but 13 is prime. Then $m+n=13$, $m-n=1$, and $m=7$, $n=6$, $m^2+n^2=49+36=85$. Hence (E).

Answer：(E)

10. A palindrome number is a positive integer that is the same when read forwards or backwards. The numbers 909 and 1221 are examples of palindromes. How many three-digit palindrome numbers are divisible by 9?

(A) 10 (B) 12 (C) 15 (D) 20 (E) 24

【Solution 1】

Suppose the palindrome number divisible by 9 is $\overline{aba}$, where $1\le a\le9$, $0\le b\le9$. A number is divisible by 9 if and only if sum of its digits is divisible by 9, vice versa.

Then $a+b+a=2a+b$ is divisible by 9.

When $2a+b=27$, we have only 999.

When $2a+b=18$, we have 585, 666, 747, 828, 909.

When $2a+b=9$, we have 171, 252, 333, 414.

There are totally $1+5+4=10$ of them. Hence (A).

【Solution 2】

Suppose the palindrome number divisible by 9 is $\overline{aba}$, where $1 \le a \le 9$, $0 \le b \le 9$. A number is divisible by 9 if and only if sum of its digits is divisible by 9, vice versa.

When $a = 1$, we have only 171.

When $a = 2$, we have only 252.

When $a = 3$, we have only 333.

When $a = 4$, we have only 414.

When $a = 5$, we have only 585.

When $a = 6$, we have only 666.

When $a = 7$, we have only 747.

When $a = 8$, we have only 828.

When $a = 9$, we have 909 and 999.

There are totally 10 of them. Hence (A).

Answer：(A)

11. The greatest common divisor of n and 24 is 2, while the greatest common divisor of $n+1$ and 24 is 3. Which of the following numbers cannot be n?

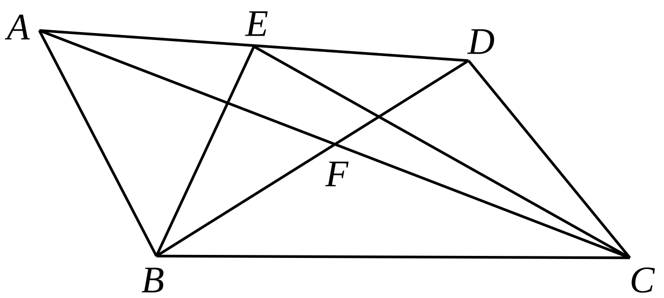
(A) 2　　(B) 14　　(C) 20　　(D) 38　　(E) 50

【Solution】

It is known that n is divisible by 2 but not by 4, only (C) not satisfied. Plug in other values to check that the conditions are satisfied. Hence (C).

Answer：(C)

12. In the figure below, let point E be the midpoint of AD and point F be the midpoint of AC. If the area of triangle ABF is 8 cm^2 and area of ADF is 6 cm^2, then what is the area, in cm^2, of triangle BCE?



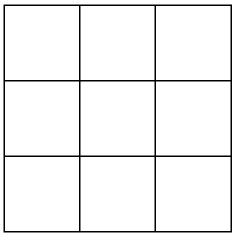
(A) 12　　(B) 13　　(C) 14　　(D) 15　　(E) 16

【Solution】

The area of BFC is equal area of ABF, which is 8 cm^2; Area of DFC is equal to area of ADF, which is 6 cm^2. Then area of $ABCD$ is $8+8+6+6=28$ cm^2. Since E is midpoint of AD, area of ABE is half of area of ABD, which is $\frac{1}{2}(8+6)=7\,\text{cm}^2$; Area of CDE is half of area of ACD, which is $\frac{1}{2}(6+6)=6\,\text{cm}^2$. Then area of BCE is $28-7-6=15\,\text{cm}^2$. Hence (D).

Answer：(D)

13. Shade 3 unit squares on the 3×3 grid below, such that there must be two shaded squares in some row and two shaded squares in some column but it must not have three shaded squares in any row or column. Find the total number of ways in shading the figure.



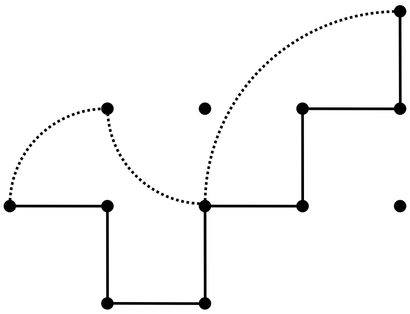
（A） 6　　（B） 18　　（C） 36　　（D） 54　　（E） 72

【Solution】

There is one black square (say A); one black square (say B) on the same row as A; one black square (say C) on the same column as A. All other squares are white. A is special since it is the only black square with black square one its same row and with black square on its same column. B is special since it is the only square on the same row as A. C is special since it is the only square on the same column as A. There are 9 ways of choosing A; 2 ways of choosing B; 2 ways of choosing C. By the principle of multiplication, there are $9\times 2\times 2=36$ ways of coloring. Hence (C).

Answer：（C）

14. In the figure below, the eight line-segments drawn are all equal to 1 m and the three dotted lines are all quarter arcs. What the is difference, in m, between the total length of all the line segments and total length of all the dotted arcs? (Use $\pi=3.14$)



（A） 0.28　　（B） 0.72　　（C） 1.28　　（D） 1.72　　（E） 4.86

【Solution】

Total length of line segments is 8 m, total length of dotted arcs is $\frac{2\pi}{4}+\frac{2\pi}{4}+\frac{2\times 2\pi}{4}=2\pi=6.28$ m, the difference is $8-6.28=1.72$ m. Hence (D).

Answer：（D）

15. Starting from $\frac{3}{4}$, add 2 to the numerator or add 3 to the denominator for each operation, but not both, and no reduction is performed. At least how many operations one needs to get a fraction again that is of the same value as $\frac{3}{4}$?

（A） 13　　（B） 17　　（C） 20　　（D） 26　　（E） 34

【Solution】

Suppose the numerator is added a times, and denominator is added b times. Then $\frac{3+2a}{4+3b}=\frac{3}{4}$, which gives $8a=9b$. The least values of a, b are $a=9$, $b=8$, the least number of operations is then $9+8=17$. Hence (B).

Answer：(B)

16. Let a, b, c and d be consecutive positive integers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{36}+\frac{1}{45}=1$. What is the value of $a+b+c+d$?

(A) 10　　(B) 12　　(C) 14　　(D) 16　　(E) 18

【Solution】

One has that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=1-\frac{1}{45}-\frac{1}{36}=\frac{19}{20}$. Since a, b, c, d are consecutive positive integers, then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}<\frac{1}{a}+\frac{1}{a}+\frac{1}{a}+\frac{1}{a}=\frac{4}{a}$. Then $\frac{4}{a}>\frac{19}{20}$, $a<4\frac{4}{19}$.

Moreover $\frac{19}{20}<1$, then $a>1$. We get $a=2$, 3 or 4.

If $a=2$, $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\frac{30}{60}+\frac{20}{60}+\frac{15}{60}+\frac{12}{60}=\frac{77}{60}>1>\frac{19}{20}$, not a solution;

If $a=3$, $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{20}{60}+\frac{15}{60}+\frac{12}{60}+\frac{10}{60}=\frac{57}{60}=\frac{19}{20}$, this is a solution;

If $a=4$, $\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}<\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{19}{20}$, not a solution.

Then $a=3$, $b=4$, $c=5$, $d=6$, $a+b+c+d=3+4+5+6=18$. Hence (E).

Answer：(E)

17. Replace all the 9 variables in the expression $a+\frac{c}{b}+d+\frac{f}{e}+g+\frac{i}{h}$ using the digits 1, 2, 3, ⋯ 9, where each digit is only used once. Find the maximum possible value of the result.

(A) 25　　(B) $31\frac{2}{3}$　　(C) $33\frac{2}{3}$　　(D) $33\frac{5}{6}$　　(E) $34\frac{1}{6}$

【Solution】

It is obvious that b, e, h should be the smallest 3 numbers, say $b=1$, $e=2$, $h=3$, then

$$a+\frac{c}{b}+d+\frac{f}{e}+g+\frac{i}{h}=a+c+d+g+\frac{f}{2}+\frac{i}{3}=a+c+d+g+f+i-(\frac{f}{2}+\frac{2i}{3}).$$

Then one should choose $i=4$, $f=5$, and any choice of remaining 4 numbers. Then result is $33\frac{5}{6}$. Hence (D).

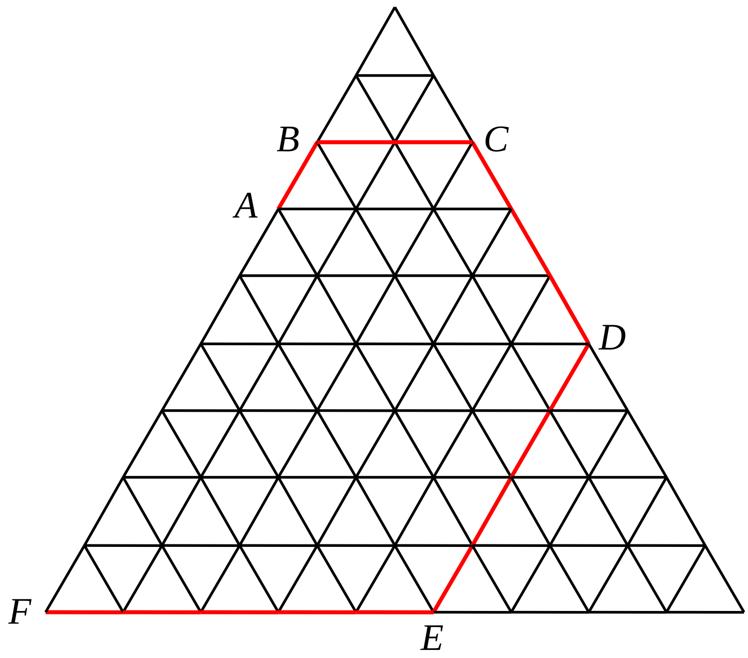
Answer：(D)

18. An ant crawls on the plane. It starts at point A and crawls for 1 cm, then turns $60°$ to the right; crawls for another 2 cm and again turns $60°$ to the right; crawls for another 3 cm and turns $60°$ to the right; crawls for another 4 cm and turns $60°$ to the right; and finally it craws 5 cm and reaches F. What is the distance, in cm, between A and F?

（A） 0 （B） 3 （C） $3\sqrt{3}$ （D） 6 （E） $6\sqrt{3}$

【Solution】

Put the route of the ant onto an equilateral network, and find the distance between A and F is 6 cm. Hence (D).



Answer：（D）

19. Arrange all proper fractions in a sequence such that the denominators are all in non-decreasing order, and while for equal denominators, the numerator is arranged in increasing order. The resulting sequence is as follows:

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \cdots.$$

It is known that the sum of first n terms of this sequence is an integer, which of the integers below is a possible value for n?

（A） 2015 （B） 2016 （C） 2017 （D） 2018 （E） 2019

【Solution】

Since $2016=1+2+3+\cdots+63$, the first 2016 terms sum to

$$\frac{1}{2}+(\frac{1}{3}+\frac{2}{3})+(\frac{1}{4}+\frac{2}{4}+\frac{3}{4})+\cdots+(\frac{1}{64}+\frac{2}{64}+\cdots+\frac{63}{64})=\frac{1}{2}+\frac{2}{2}+\frac{3}{2}+\cdots+\frac{63}{2}=1008.$$

All other options are incorrect, then n can only be 2016 among these options. Hence (B).

Answer：（B）

20. Using the digits 1, 2, 3, 4, 5, 6, 7 and 8 only once, create a sequence such that there is one number between 1 and 2, two numbers between 2 and 4, three numbers between 3 and 6 and four numbers between 4 and 8. How many different ways can we do this?

（A）12 （B）24 （C）36 （D）48 （E）60

【Solution】

First notice that 2 and 8 cannot be on different side of 4, otherwise there will be $2+1+4=7$ numbers between 2 and 8, a contradiction.

If 2 and 8 are both to the right of 4, the sequence partially looks like 4*ab*2*c*8, where 1 must be *a*.

If 4 is the first from the left, then the sequence looks like 41*b*2*c*8*de*, 3 and 6 must be at *b*, *d*, two choices for them and two choices for remaining 5 and 7;

If 4 is the second from the left, it looks like *d*41*b*2*c*8*e*, 3 and 6 must be at *b*, *e*; two choices for them and two choices for 5 and 7;

If 4 is the third from the left, it looks like *de*41*b*2*c*8, 3 and 6 are at *b*, *d*, two choices for them and two choices for 5 and 7;

When 2 and 8 are both to the left of 4, it is symmetric to above arguments.

The total number is then $(2\times2+2\times2+2\times2)\times2=24$. Hence (B).

Answer：（B）

21. An integer is known to be both a multiple of 3 and 7. Among all its divisors, there is one more multiple of 7 than multiple of 3. What is the least possible integer that satisfies the condition?

【Solution 1】

Among its divisors, those which are not multiples of 7 is one less than those which are not multiples of 3.

Write the integer as $3^{\alpha}7^{\beta}p_1^{a_1}\cdots p_k^{a_k}$, there are $(\alpha+1)(a_1+1)\cdots(a_k+1)$ of its divisors which is not a multiple of 7; there are $(\beta+1)(a_1+1)\cdots(a_k+1)$ of its divisors which is not a multiple of 3. Then $(\beta-\alpha)(a_1+1)\cdots(a_k+1)=1$, so $k=0$, $\beta=\alpha+1$, the least such integer is $3\times7^2=147$.

【Solution 2】

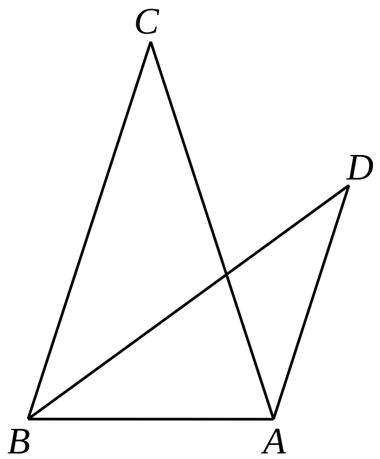
If the number is not a multiple of 7^2, then exactly half of its divisors is divisible by 7; It also has at least half number of divisors which is divisible by 3, a contradiction. The number is a multiple of 7^2, then this number is at least $3\times7^2=147$, 147 satisfies the condition by direct check.

【Solution 3】

It is obvious the number is a multiple of 21, direct check shows that 147 is the least such number.

Answer：147

22. In the figure below, it is known that $BC//AD$, $BC = AC$, $BA = AD$ and $\angle C = \angle D$. Find the measure, in degrees, of $\angle BAC$?



【Solution】

By the condition $\angle CBA = \angle CAB$, $\angle CBD = \angle BDA = \angle ABD$. If $\angle C = \angle D = x$, then $\angle CAB = \angle ABC = 2x$, $x + 2x + 2x = 180°$, solve to get $x = 36°$, so $\angle BAC = 2x = 72°$.

Answer：072

23. Let a, b and c be real numbers such that $abc = 1$ and $a + b + c = ab + bc + ca = 6$. What is the value of $a^3 + b^3 + c^3$?

【Solution】

$(a-1)(b-1)(c-1) = abc - (ab + bc + ca) + (a + b + c) - 1 = 0$.

Then one of a, b, c is 1, if $a = 1$, then $bc = 1$, $b + c = 5$.

We get $a^3 + b^3 + c^3 = 1 + (b + c)((b + c)^2 - 3bc) = 111$.

Answer：111

24. Let a be a positive integer such that $2018 - a^2$ is also positive. What is the maximum possible number of divisors of $2018 - a^2$?

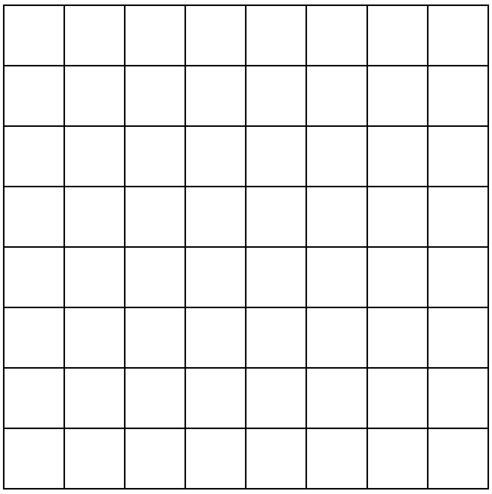
【Solution】

Since 2018 has remainder 2 divided by 4; remainder 2 divided by 3 and remainder 3 divided by 5. And no square has remainder 2 divided by 4; or remainder 2 divided by 3; or remainder 3 divided by 5. $2018 - a^2$ is divisible by at most 1st power of 2 and no divisors of 3 or 5. Since $7^4 > 2018 > 2018 - a^2$, $2018 - a^2$ has at most three odd prime divisors (repetitions counted). So $2018 - a^2$ can have at most $2^4 = 16$ different divisors.

When $a = 4$, $2018 - a^2 = 2002 = 2 \times 7 \times 11 \times 13$ has 16 different divisors. 16 is the maximum.

Answer：016

25. Cut the 8×8 square table below into rectangles along grid lines such that no two rectangles are identical. What is the maximum number of rectangles one can get? (Note: A square is considered a rectangle.)

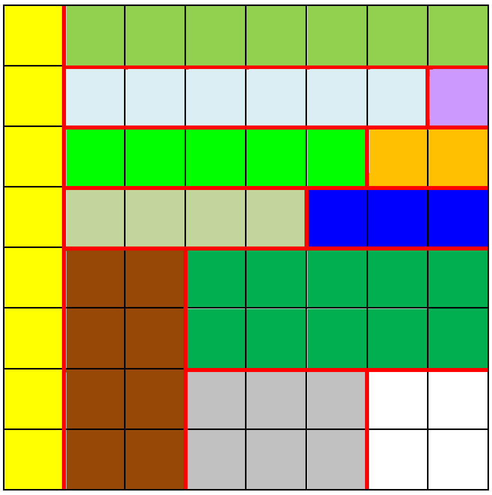


【Solution】

For more rectangles, the rectangles we get need to be as small in area as possible. There one rectangle of area 1, 2 or 3; two rectangles of area 4(1×4 and 2×2); one of area 5; two of area 6(1×6 and 2×3); one of area 7 (1×7); two of area 8 (1×8 and 2×4); two of area 9 (1×9(but it can't exist) and 3×3). These are the first 12 rectangles of smallest area. Since

$$1+2+3+4+4+5+6+6+7+8+8+9=63<8\times 8=64,$$
$$8\times 8=64<1+2+3+4+4+5+6+6+7+8+8+9+9=72,$$

It shows that there are at most 12 different rectangles. The figure below shows an example:



Answer：012