Solution Key to Second Round of IMAS 2018/2019
Upper Primary Division

1. How many different prime divisors does $(2019-2018)\times(2019-2017)\times\cdots\times(2019-2012)\times(2019-2011)$ have?

（A）2　　（B）3　　（C）4　　（D）5　　（E）8

【Solution】

$$(2019-2018)\times(2019-2017)\times\cdots\times(2019-2012)\times(2019-2011)$$
$$=1\times2\times3\times4\times5\times6\times7\times8$$
$$=2\times3\times(2\times2)\times5\times(2\times3)\times7\times(2\times2\times2\times2)$$

There are 4 different divisors 2, 3, 5, 7. Hence (C).

Answer：（C）

2. Four students went mountain climbing together. They spent \$50 in total on bus fare. While at the mountain top, each of them spent \$5 on beverages. What is the average expense for each student?

（A）\$12.5　　（B）\$13.75　　（C）\$17.5　　（D）\$30　　（E）\$55

【Solution 1】

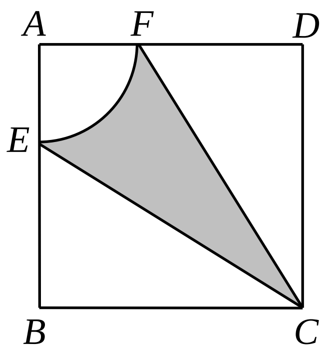
Totally they spent $\$50+5\times4=70$, so on average each spent $\$70\div4=17.5$.

【Solution 2】

Average bus fare is $\$50\div4=12.5$, so on average each spent $\$12.5+5=17.5$ in total. Hence (C).

Answer：（C）

3. In the figure below, the side length of the square *ABCD* is 8 cm and the radius of the sector *AEF* is 3 cm. What is the area, in cm^2, of the shaded region? (Take π as 3.14, and round off to TWO decimal places)



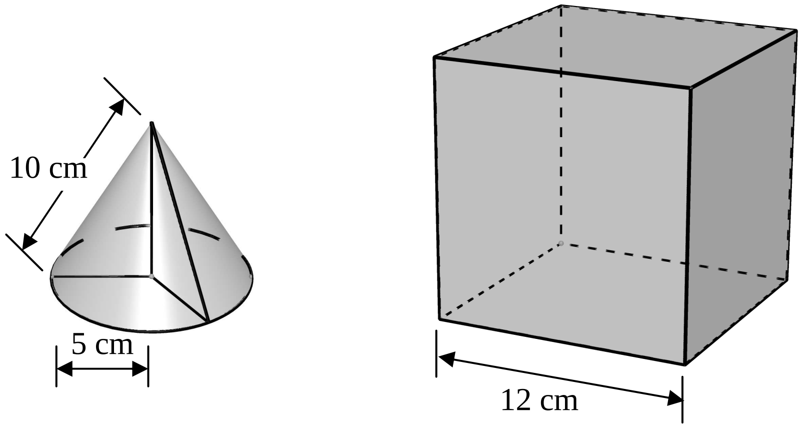
（A）16.94　　（B）19.38　　（C）24.38　　（D）26.94　　（E）31.07

【Solution】

As in the figure, the area of the shaded area equals area of the square minus area of the sector and two right triangles. Since $BE=8-3=5$ cm. Thus the shaded area is $8\times8-\frac{1}{4}\times3.14\times3^2-2\times\frac{1}{2}\times8\times5\approx16.94$ cm^2. Hence (A).

Answer：（A）

4. In the figure below, a circular cone and a cube are shown. If a solid shape is formed by attaching the two shapes together, what is the least surface area, in cm^2, of the resulting shape? (Take π as 3.14,, and round off to ONE decimal place)



（A）785.5　（B）942.5　（C）1000.5　（D）1021.0　（E）1099.5

【Solution】

To make the surface area as small as possible, the overlapping area should be as large as possible. For the circular cone, if the side is glued with the cube, the total surface area is not reduced. Hence the bottom of the cone should be glued to the cube. Since the diameter of the bottom of the cone is $5\times2=10$ cm, but side length of the cube is 12 cm, which is larger, the whole bottom face of the cone can be glued to part of a face of the cube. The total surface area of the resulted shape is at least

$12\times12\times6+\frac{1}{2}\times3.14\times5\times2\times10-3.14\times5\times5=942.5\,cm^2$.　Hence (B).

Answer：（B）

5. Given four distinct non-zero digits a, b, c and d, if $\overline{ab}+\overline{cd}=\overline{dc}+\overline{ba}$, then this expression is called a palindrome expression and the sum of the two numbers $\overline{ab}+\overline{cd}$ is called a palindrome sum. For example, $53+46=64+35=99$. What is the minimum possible value of a palindrome sum?

（A）22　（B）33　（C）44　（D）55　（E）99

【Solution】

Since $\overline{ab}+\overline{cd}=\overline{dc}+\overline{ba}$, it is known that $10(a+c)+(b+d)=10(b+d)+(a+c)$, thus $a+c=b+d$. The least number that can be represented as the sum of two different positive numbers in two different ways is $5=1+4=2+3$, thus the least palindrome sum is 55, for example, $12+43=34+21=55$.　Hence (D).

Answer：（D）

6. There are a total of 40 students in a class. 23 of them are able to ride bikes, 33 of them are able to swim and 5 of them are unable to do either. How many students in this class are able to ride bikes but are not able to swim?

【Solution】

The number of students who are able to do at least one sport is $40-5=35$. Then the number of students who can do both sports is $23+33-35=21$. The number of students who can ride but not swim is then $23-21=2$ students.

Answer：2 students

7. One day, Adam drove from Town *A* to Town *B* at a speed of 60 km/h. After an hour, the car stopped because of a breakdown, and because of this, Adam immediately called Bob for help. Bob then drove from *A* along the same route at a speed of 80 km/h. When Bob met Adam, he towed Adam's car to *B*, at a speed of 40 km/h. The distance between *A* and *B* is 180 km. How long did Adam spend travelling for the whole trip?

【Solution】

It took Bob $60 \div 80 = \frac{3}{4}$ hours to arrive at the place where Adam stopped. Then it took $(180 - 60 \times 1) \div 40 = 3$ hours for both cars to arrive at *B*. In total, it took Adam $1 + \frac{3}{4} + 3 = 4\frac{3}{4} = 4.75$ hours from *A* to *B*.

Answer： $4\frac{3}{4} = 4.75$ hours

8. How many different prime numbers a are there such that $a+20$ and $a+40$ are also prime numbers?

【Solution 1】

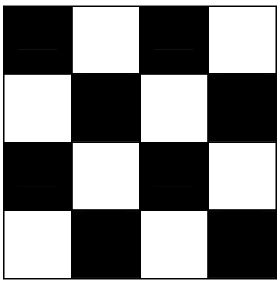
Since 20 and 40 are both even numbers and $a+20$ and $a+40$ are primes, $a \neq 2$. Since the remainder of 20 divided by 3 is 2, the remainder of *a* divided by 3 can not be 1, otherwise $a+20$ is divisible by 3 and hence it is a composite number. This is a contradiction. Similarly, the remainder of *a* divided by 3 can not be 2, otherwise $a+40$ is a composite number. This is also a contradiction. So *a* is divisible by 3 and prime, which must be $a=3$, there is one possible value of *a*.

【Solution 2】

Regardless of the value of *a*, one of the three numbers *a*, *a* + 20, and *a* + 40 must be divisible by 3. From the given information, we known that *a* is a prime number, and *a* + 20 and *a* + 40 are also prime numbers. Therefore, *a* must be divisible by 3 and a prime number, it follows only *a* = 3, that is, there is one possible value of *a*.

Answer：1

9. Four identical chess pieces are to be placed into a 4×4 chess board that is colored black and white alternately, as shown in the figure below. You can place at most one chess piece on each square. All chess pieces must be placed in squares of the same color and no two pieces are on the same row or on the same column. In how many different ways can the chess pieces be placed?

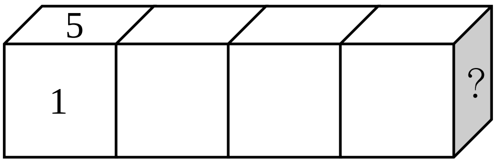


【Solution】

If all pieces were placed into black squares, there are two ways to put the pieces on the first and the third row, two ways to put the pieces on the second and the fourth row, totally there are 4 ways. Similarly, there are 4 ways to place all pieces into white squares. Hence there are 8 ways in total.

Answer：8 ways

10. The numbers 1, 2, 3, 4, 5 and 6 are written on the six faces of a unit cube without repetition. Each face contains one number and the sum of the numbers in every two opposite faces is 7. Put four such cubes side by side as shown in the figure below, such that sum of every two numbers of every two touched faces is 8. Find the number marked with "?" in the figure.



【Solution】

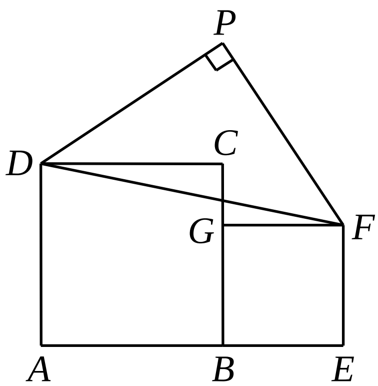
Since sum of numbers on opposite faces is 7, then 1 is opposite to 6, and 2 is opposite to 5, and 3 is opposite to 4. The right face of the first cube has number 3 or 4.

If it is 3, the left face of the second cube has number 5, right face has number 2, left face of the third cube has number 6, right face has 1, left face of the fourth cube has number 7, which is a contradiction.

If it is 4, the left face of the second cube has number 4, right face has number 3, left face of the third cube has number 5, right face has 2, left face of the fourth cube has number 6, right face has number 1, which is the place with mark "?".

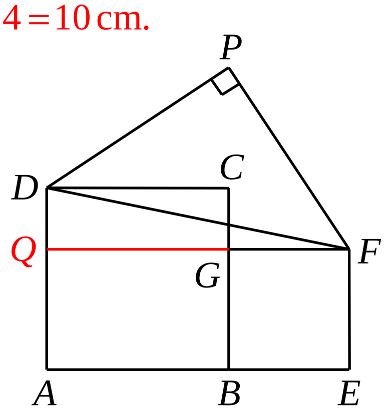
Answer：1

11. In the figure below, the side lengths of squares *ABCD* and *BEFG* are 6 cm and 4 cm respectively and triangle *DFP* is an isosceles right triangle. What is the area, in cm^2, of triangle *DFP*?



【Solution】

Extend *FG* to intersect *AD* at point *Q*. Then triangle *DQF* is a right triangle, and $DQ = 6 - 4 = 2$ cm, $QF = 6 + 4 = 10$ cm.



By Pythagorean theorem $DP^2 + PF^2 = DF^2 = DQ^2 + QF^2 = 2^2 + 10^2 = 104$. Since $DP = PF$, then $DP^2 = 52$. Area of triangle *DFP* is $\frac{1}{2} \times DP \times PF = \frac{1}{2} \times DP^2 = 26$ cm^2.

Answer：26 cm^2

12. A mouse starts from the top left-most unit square marked with "*I*", follows a route to form the word "*IMAS2019*" by moving from one square to another square that share a common side. How many different routes of eight squares are there?

I	*M*	*A*	*S*	
M	*A*	*S*	*2*	*0*
A	*S*	*2*	*0*	*1*
S	*2*	*0*	*1*	*9*
	0	*1*	*9*	

【Solution】

In the following table, each square is filled with the number of routes reaching it. The number can be derived by recursion: each square is filled with sum of the numbers in adjacent squares with previous marks. From the table, it shows that the number of different routes with length eight is $34+34=68$.

1	1	1	1	
1	2	3	4	4
1	3	6	10	14
1	4	10	20	34
	4	14	34	

Answer： 68 routes

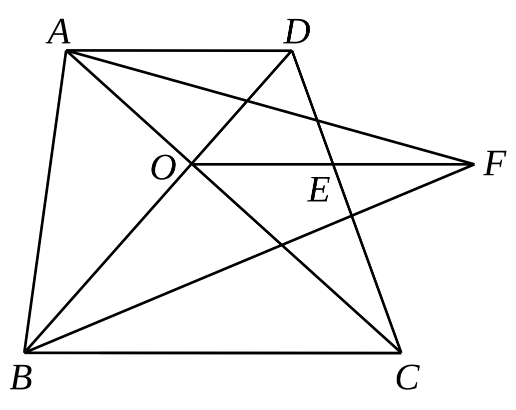
13. If a and b are positive integers such that $1 \le a < b \le 60$ and $a \times b$ is divisible by 5. How many different ordered pairs of (a, b) are there?

【Solution】

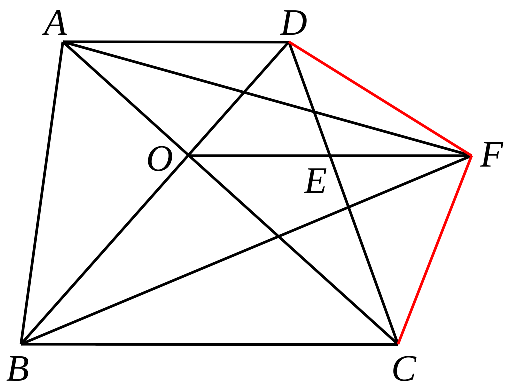
Among positive integers less than or equal to 60, there are 12 numbers of them that are divisible by 5 and 48 numbers that are not. If both a and b are divisible by 5, one needs to choose 2 numbers among 12, so there are $\frac{12\times 11}{2}=66$ pairs. If only one of a, b is divisible by 5, and another is not divisible by 5, there are $12\times 48=576$ such pairs. In total, there are $66+576=642$ pairs.

Answer：642 pairs

14. In the figure, *ABCD* is a trapezoid, with side *AD* that is parallel to *BC*, diagonals *AC* and *BD* intersect at point *O*, and line *OE* is parallel to *BC* and intersects *CD* at point *E*. Now, extend *OE* to point *F* such that $OE = EF$. If $AD = 6$ cm, $BC = 10$ cm and the area of trapezoid *ABCD* is 64 cm^2, what is the area, in cm^2, of triangle *ABF*?



【Solution 1】



Connect *DF* and *CF*, by properties of parallel lines one knows that $S_{\triangle AOF}=S_{\triangle DOF}$, $S_{\triangle BOF}=S_{\triangle COF}$, then $S_{AOBF}=S_{DOCF}$. *E* is midpoint of *OF*, then $S_{\triangle DOC}=S_{\triangle DCF}$, and $S_{DOCF}=2S_{\triangle DOC}$. **(5 points)**

Since $\frac{S_{\triangle ACD}}{S_{\triangle ABC}}=\frac{S_{\triangle ABD}}{S_{\triangle BCD}}=\frac{AD}{BC}=\frac{6}{10}=\frac{3}{5}$, $S_{\triangle ABD}=\frac{3}{5+3}S_{ABCD}=24\text{ cm}^2$. **(5 points)**

Since triangle *DOA* and triangle *BOC* are similar, we have $\frac{DO}{OB}=\frac{AD}{BC}=\frac{6}{10}=\frac{3}{5}$.

(Or by common side theorem, $\frac{DO}{OB}=\frac{S_{\triangle ACD}}{S_{\triangle ABC}}=\frac{3}{5}$, then $\frac{S_{\triangle ADO}}{S_{\triangle ABO}}=\frac{DO}{OB}=\frac{3}{5}$.)

Hence $S_{\triangle ABO}=\frac{5}{3+5}S_{\triangle ABD}=15\text{ cm}^2$. **(5 points)**

As *AD* is parallel to *BC*, $S_{\triangle ABD}=S_{\triangle ACD}$, subtract $S_{\triangle AOD}$ on both sides one has $S_{\triangle ABO}=S_{\triangle DOC}$.

Hence $S_{\triangle ABF}=S_{\triangle ABO}+S_{AOBF}=S_{\triangle ABO}+S_{DOCF}=S_{\triangle ABO}+2S_{\triangle DOC}=3S_{\triangle ABO}=45\text{ cm}^2$.

(5 points)

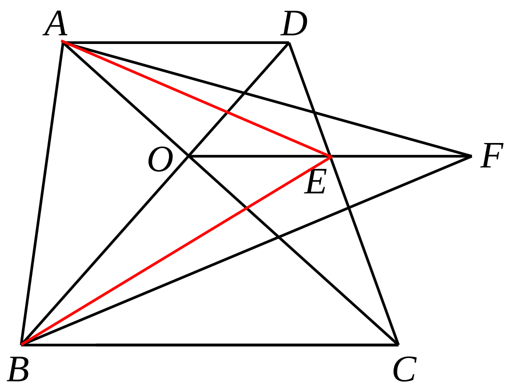
【Solution 2】

As *AD* is parallel to *BC*, triangle *AOD* and *COB* are similar. Thus we have

$$\frac{DO}{OB}=\frac{AO}{OC}=\frac{AD}{BC}=\frac{6}{10}=\frac{3}{5} \text{ and } \frac{S_{\triangle AOD}}{S_{\triangle COB}}=\frac{AD^2}{BC^2}=\left(\frac{6}{10}\right)^2=\frac{9}{25}.$$ **(5 points)**

Let the area of triangle *AOD* be 9*x*, then the area of triangle *COB* would be 25*x*. Then the areas of triangle *AOB* and *COD* are both $9x\times\frac{5}{3}=15x$. So the area of trapezoid *ABCD* is $9x+25x+15x+15x=64x$, i.e. $x=1$.Thus the areas of triangle *AOB* and *COD* are both 15 cm^2. **(5 points)**

Now, connect *AE* and *BE*. Since $OE=EF$, the areas of triangles *DOE*, *AOE* and *AEF* are all equal by the property of parallel lines. Thus the area of triangle *AOF* is twice of the area of triangle *DOE*. Similarly, the areas of triangles *COE*, *BOE* and *BEF* are all equal. Thus the area of triangle *BOF* is twice of the area of triangle *COE*. **(5 points)**



Observe that the area of triangle *ABF* is equal to the sum of areas of triangle *ABO*, *AOF* and *BOF*, which is $15+2\times(S_{\triangle DOE}+S_{\triangle COE})=15+2\times15=45$ cm^2. **(5 points)**

Answer： 45 cm^2

15. A robot can generate a set of digit codes according to user's reasonable instructions. Wayne gives out the following commands:
    (1) Each code is a three-digit number (nonzero for the left-most digit).
    (2) Every two codes in the set have identical digits at no more than one corresponding positions.

    Find the maximum number of codes in a set the robot can generate.

【Solution 1】

From the given information, any two codes are different in at least two digits. Hence, the number of codes cannot exceed 90. Since, the hundred-digit can only be 9 digits from 1 to 9, the ten-digit can be 10 possible digits from 0 to 9, so the first two digits in the 3-digit code number can form a total of $9\times10=90$ different two digits. If the total number of codes is more than 90, then by pigeon-hole principle, at least two codes have the same hundred-digit and the same ten-digit, which is a contradiction! (5 points)

Construct 90 codes as follows: the first two digits take 10 to 99, once for each; the third digit equals the last digit of sum of first two digits. (5 points)

Next, we show that such set of codes satisfies the command. For any code $\overline{abd}$, $a+b-d$ is divisible by 10, thus each of *a*, *b*, *d* is determined by the other two. If two codes have identical digits at two corresponding places, their digits at the third place are also the same, showing that they are the same code. Thus every two codes have identical digits at no more than one places. (10 points)

(5 points)

【Solution 2】

List down correctly all the 90 codes satisfying the command. (10 points, any error or missing results in 0 points)

Prove that there are at most 90 codes. (10 points)

Answer：90 codes

【Note】

We also can construct 90 codes as follows: the first two digits take 10 to 99, once for each; the third digit satisfies that the sum of three digits is a multiple of 10. If two codes have one identical digits and the other one is different in the corresponding position, then the third digits are also different.