Chapter 10 Surds (Radicals)

10.1 Surds

The square root of *a*, $\sqrt{a}\,(a \geq 0)$ where *a* appears under a square root sign (or radical sign) is called a Surd (or a radical). For example, $\sqrt{3}$, $\sqrt{\frac{3}{5}}$, $\sqrt{b^2+1}$, $\sqrt{(a-b)^2}$ are surds. In the doman of real numbers, we cannot take square root of negative numbers. Therefore, $\sqrt{-5}$ and $\sqrt{a}\,(a<0)$ are undefined. In this chapter, all alphabets are positive numbers unless otherwise specified.

As mentioned in the previous chapter, the positive square root of a positive number a is also called the principal square root, labelled as $\sqrt{a}$. This indicates that $\sqrt{a}$ is a positive number and $\sqrt{a}>0$. Sqaure root of 0 is also called the principal square root of zero, labelled as $\sqrt{0}$ and we have $\sqrt{0}=0$. According to the above analysis, it can be observed that $\sqrt{a} \geq 0\,(a \geq 0)$, that is $\sqrt{a}\,(a \geq 0)$ always a non-negative number (a non-negative number is either zero or positive.)

According to the definition of square root, if a number when squared equals 2, then the number is called the square root of 2. Therefore, we have the relaltionship that the square of the square root of 2 is 2:

$$(\sqrt{2})^2 = 2.$$

In general, if a number when squared equals *a*, then the number is called the square root of *a*. Therefore, we have the relationship that the square of the square root of *a* is *a*:

$$\boxed{(\sqrt{\boldsymbol{a}})^2 = \boldsymbol{a}\ \ (\boldsymbol{a} \geq \boldsymbol{0})}$$

【Example 1】 Calculate:

(1) $(\sqrt{4})^2$; (2) $(\sqrt{5})^2$;

(3) $\left(\sqrt{\frac{3}{5}}\right)^2$; (4) $(2\sqrt{3})^2$.[3]

Solution (1) $(\sqrt{4})^2 = 2^2 = 4$ or $(\sqrt{4})^2 = 4$;

(2) $(\sqrt{5})^2 = 5$;

(3) $\left(\sqrt{\frac{3}{5}}\right)^2 = \frac{3}{5}$;

(4) $(2\sqrt{3})^2 = 2^2 \times (\sqrt{3})^2 = 4 \times 3 = 12$.

Reversing the above formula $(\sqrt{a})^2 = a\,(a \geq 0)$ will result in:

$$\boldsymbol{a} = (\sqrt{\boldsymbol{a}})^2 \quad (a \geq 0).$$

By making use of this formula, we can rewrite any non-negative number in the form of a square of another number.

【Example 2】 Rewrite the following non-negative number in the form of a square:

(1) 2; (2) 0.5; (3) $\frac{1}{7}$; (4) *ab*.

Solution (1) $2 = (\sqrt{2})^2$; (2) $0.5 = (\sqrt{0.5})^2$;

(3) $\frac{1}{7} = \left(\sqrt{\frac{1}{7}}\right)^2$; (4) $ab = (\sqrt{ab})^2$.

Practice

1. Calculate:

(1) $(\sqrt{0.5})^2$; (2) $\left(\sqrt{\frac{2}{7}}\right)^2$; (3) $(5\sqrt{7})^2$; (4) $\left(-3\sqrt{\frac{1}{3}}\right)^2$.

[3] $2\sqrt{3}$ means $2 \times \sqrt{3}$. In general, $b\sqrt{a}$ means $b \times \sqrt{a}$.

Practice

2. Rewrite the following non-negative number in the form of a square:

(1) 9; (2) 6; (3) 2.5; (4) 0.25; (5) b; (6) $4a$.

We now examine the principal square root of a^2, i.e. $\sqrt{a^2}$, when a takes on different values, namely $a>0$, $a=0$ and $a<0$.

(1) $\sqrt{2^2}=2$, $\sqrt{3^2}=3$;

In general, when $a>0$, $\sqrt{a^2}=a$.

(2) $\sqrt{0^2}=0$;

In other words, when $a=0$, $\sqrt{a^2}=a$.

(3) $\sqrt{(-2)^2}=\sqrt{4}=2=-(-2)$ since 2 and -2 are opposite numbers;

$\sqrt{(-3)^2}=\sqrt{9}=3=-(-3)$ since 3 and -3 are opposite numbers;

In general, when $a<0$, $\sqrt{a^2}=-a$.

Summarising the above, we have:

$$\sqrt{a^2}=\begin{cases} a & (a>0) \\ 0 & (a=0) \\ -a & (a<0) \end{cases}$$

and it is known that

$$|a|=\begin{cases} a & (a>0) \\ 0 & (a=0) \\ -a & (a<0) \end{cases}$$

By comparing $\sqrt{a^2}$ and $|a|$, we have:

$$\boxed{\sqrt{a^2}=|a|=\begin{cases} a & (a>0) \\ 0 & (a=0) \\ -a & (a<0) \end{cases}}$$

【Example 3】 Calculate:

(1) $\sqrt{(-1.5)^2}$;

(2) $\sqrt{(a-3)^2}$ ($a<3$).

Solution (1) $\sqrt{(-1.5)^2}=|-1.5|=1.5$;

(2) $\sqrt{(a-3)^2}=|a-3|$

$\because$ $a<3$

$\therefore$ $a-3<0$

$\therefore$ $\sqrt{(a-3)^2}=|a-3|=-(a-3)=3-a$

Practice

1. (*Mental*) Is the following equation correct? Why?

(1) $(\sqrt{7})^2=7$; (2) $(-\sqrt{7})^2=-7$;

(3) $\sqrt{6^2}=6$; (4) $\sqrt{(-6)^2}=-6$.

2. (*Mental*) Fnd the value of each of the following expression:

(1) $(\sqrt{0.8})^2$; (2) $\sqrt{0.8^2}$;

(3) $\sqrt{(-0.8)^2}$; (4) $-\sqrt{(-0.8)^2}$.

3. Simplify:

(1) $\sqrt{(5-9)^2}$; (2) $\sqrt{\left(3\frac{1}{2}-2\right)^2}$;

(3) $\sqrt{(b-4)^2}$ ($b>4$); (4) $\sqrt{(m-n)^2}$ ($m<n$).

Practice

4. Two persons, A and B, calculated the value of $a+\sqrt{1-2a+a^2}$. They obtained different answers when $a=5$.
   A calculated the result as:
   $$a+\sqrt{1-2a+a^2}=a+\sqrt{(1-a)^2}=a+1-a=1$$
   B calculated the result ast:
   $$a+\sqrt{1-2a+a^2}=a+\sqrt{(a-1)^2}=a+a-1$$
   $$=2a-1=2\times5-1=9$$
   Which answer is correct? For the answer that is incorrect, which part of the calculation is incorrect? Why?

10.2 Basic property of Surd

As we understand, the surd $\sqrt{a}$ ($a\geq0$) is the principal square root of *a*. Therefore, when we study the property of Surd, it is suffice to study the property of the principal square root.

1. Product of principal square roots

Let us examine the following example:

$$(\sqrt{4\times9})^2=4\times9=36$$

$$(\sqrt{4}\times\sqrt{9})^2=(\sqrt{4})^2\times(\sqrt{9})^2=4\times9=36$$

From the first equation, we know that $\sqrt{4\times9}$ is a square root of 36. Since $\sqrt{4\times9}$ is positive, it is the principal square root of 36.

From the second equation, we know that $\sqrt{4}\times\sqrt{9}$ is a square root of 36. Since $\sqrt{4}\times\sqrt{9}$ is positive, it is the prinicipal square root of 36.

As the principal square root of a number is unique, so $\sqrt{4\times9}$ and $\sqrt{4}\times\sqrt{9}$ must be the same. That is

$$\sqrt{4\times9}=\sqrt{4}\times\sqrt{9}.$$

In general, we have:

$$\boxed{\sqrt{ab}=\sqrt{a}\bullet\sqrt{b}\quad(a\geq0\ ,\ b\geq0)}$$

That is to say, **the principal square root of a product is equal to the product of the principal square roots of the constituent factors.**

【**Example 1**】Calculate:

(1) $\sqrt{16\times81}$; (2) $\sqrt{0.09\times0.25}$;

(3) $\sqrt{17^2-8^2}$.

Solution (1) $\sqrt{16\times81}=\sqrt{16}\times\sqrt{81}=4\times9=36$;

(2) $\sqrt{0.09\times0.25}=\sqrt{0.09}\times\sqrt{0.25}=0.3\times0.5=0.15$;

(3) $\sqrt{17^2-8^2}=\sqrt{(17+8)(17-8)}=\sqrt{25}\times\sqrt{9}=5\times3=15$.

Practice

1. Calculate:
   (1) $\sqrt{49\times121}$; (2) $\sqrt{81\times169}$; (3) $\sqrt{9\times25\times225}$;
   (4) $\sqrt{26^2-10^2}$; (5) $\sqrt{0.65^2-0.16^2}$; (6) $\sqrt{25a^4b^6c^2}$.
2. Is each of the following equation correct? Why?
   (1) $\sqrt{3^2+4^2}=3+4=7$; (2) $\sqrt{41^2-40^2}=41-40=1$.

【**Example 2**】Simplify:

(1) $\sqrt{10^2\times2}$; (2) $\sqrt{48}$;

(3) $\sqrt{4a^2b^3}$; (4) $\sqrt{x^4+x^2y^2}$.

Solution (1) $\sqrt{10^2\times2}=\sqrt{10^2}\times\sqrt{2}=10\sqrt{2}$;

(2) $\sqrt{48}=\sqrt{4^2\times3}=\sqrt{4^2}\times\sqrt{3}=4\sqrt{3}$;

(3) $\sqrt{4a^2b^3}=\sqrt{2^2\bullet a^2\bullet b^2\bullet b}=2ab\sqrt{b}$;

(4) $\sqrt{x^4+x^2y^2}=\sqrt{x^2(x^2+y^2)}=x\sqrt{x^2+y^2}$.

It can be observed from Example 2 that, for any squared factors inside the radical sign, we can take the principal square root of them and move them outside the radical sign. On the other hand, we can also transform any non-negative factor(s) from the outside of the radical sign to the insdie of the radical sign by squaring them.

【Example 3】 For the following expression, move the factors from the outside of the radical sign to the inside of the radical sign without changing the value of the expression:

(1) $5\sqrt{3}$; (2) $-3\sqrt{a}$; (3) $4b\sqrt{bc}$.

Solution (1) $5\sqrt{3}=\sqrt{5^2\times 3}=\sqrt{75}$;

(2) $-3\sqrt{a}=-\sqrt{3^2\bullet a}=-\sqrt{9a}$;

(3) $4b\sqrt{bc}=\sqrt{(4b)^2\bullet bc}=\sqrt{16b^3c}$.

Think for a while: Why is it not correct to write $-3\sqrt{a}=\sqrt{(-3)^2a}=\sqrt{9a}$?

【Example 4】 For the following expression, move the factors from the outside of the radical sign to the inside of the radical sign without changing the value of the expression: (1) $10\sqrt{0.1}$; (2) $5\sqrt{\frac{1}{5}}$.

Solution (1) $10\sqrt{0.1}=\sqrt{10^2\times 0.1}=\sqrt{10}$;

(2) $5\sqrt{\frac{1}{5}}=\sqrt{5^2\times\frac{1}{5}}=\sqrt{5}$.

Practice

1. Simplify:

(1) $\sqrt{18}$; (2) $-\sqrt{27\times 15}$; (3) $\sqrt{21^2-4^2}$;

(4) $\sqrt{9x}$; (5) $\sqrt{5a^3}$; (6) $\sqrt{8x^2y^3}$;

(7) $\frac{1}{6}\sqrt{9a^2bc^3}$; (8) $\sqrt{16(x+2)^3}$.

Practice

2. For the following expression, move the factors from the outside of the radical sign to the inside of the radical sign without changing the value of the expression:

(1) $5\sqrt{2}$; (2) $-7\sqrt{3}$; (3) $6\sqrt{5}$;

(4) $2\sqrt{0.5}$; (5) $-12\sqrt{\frac{c}{2}}$; (6) $a\sqrt{\frac{b}{a}}$.

3. (*Mental*) Is the following equation correct? Why?

(1) $2a\sqrt{b}=\sqrt{2a^2b}$; (2) $-3\sqrt{2}=\sqrt{(-3)^2\times 2}=\sqrt{18}$;

(3) $3\sqrt{\frac{a}{3}}=\sqrt{a}$.

2. Quotient of principal square roots

Let us look at the following example;

$$\left(\sqrt{\frac{2}{5}}\right)^2=\frac{2}{5}$$

$$\left(\frac{\sqrt{2}}{\sqrt{5}}\right)^2=\frac{(\sqrt{2})^2}{(\sqrt{5})^2}=\frac{2}{5}$$

From the first equation, we know that $\sqrt{\frac{2}{5}}$ is a square root of $\frac{2}{5}$. Since $\sqrt{\frac{2}{5}}$ is positive, it is the principal square root of $\frac{2}{5}$.

From the second equation, we know that $\frac{\sqrt{2}}{\sqrt{5}}$ is a square root of $\frac{2}{5}$. Since $\frac{\sqrt{2}}{\sqrt{5}}$ is positive, it is the principal square root of $\frac{2}{5}$.

As the principal square root of a number is unique, so $\sqrt{\frac{2}{5}}$ and $\frac{\sqrt{2}}{\sqrt{5}}$ are the same. That is

$$\sqrt{\frac{2}{5}}=\frac{\sqrt{2}}{\sqrt{5}}.$$

In general, we have:

$$\boxed{\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}} \quad (a \geq 0\text{，}b>0)}$$

That is to say, **the principal square root of a quotient is equal to the quotient of the principal square root of the dividend divided by the principal square root of the divisor.**

【Example 5】 Calculate:

(1) $\sqrt{\frac{4}{9}}$; (2) $\sqrt{1\frac{15}{49}}$;

(3) $\sqrt{\frac{3}{100}}$; (4) $\sqrt{\frac{25x^4}{81y^2}}$.

Solution (1) $\sqrt{\frac{4}{9}}=\frac{\sqrt{4}}{\sqrt{9}}=\frac{2}{3}$;

(2) $\sqrt{1\frac{15}{49}}=\sqrt{\frac{64}{49}}=\frac{\sqrt{64}}{\sqrt{49}}=\frac{8}{7}=1\frac{1}{7}$;

(3) $\sqrt{\frac{3}{100}}=\frac{\sqrt{3}}{\sqrt{100}}=\frac{1}{10}\sqrt{3}$;

(4) $\sqrt{\frac{25x^4}{81y^2}}=\frac{\sqrt{25x^4}}{\sqrt{81y^2}}=\frac{5x^2}{9y}$.

Let us look at another example:

$$\sqrt{\frac{a}{b}}=\sqrt{\frac{a \bullet b}{b \bullet b}}=\frac{\sqrt{ab}}{\sqrt{b^2}}=\frac{1}{b}\sqrt{ab}.$$

The above example shows a very useful transformation. There is a surd $\sqrt{b}$ in the denominator which we would like to eliminate. We obserse that, by multiplying both the numerator and the denominator by the same surd $\sqrt{b}$, the denominator is converted to $\sqrt{b^2}=b$. which does not fall under a radical sign and is no longer a surd. The expression $\sqrt{\frac{a}{b}}$ is thus simplified to $\frac{1}{b}\sqrt{ab}$ which does not have a surd in its denominator.

【Example 6】 For the following expression, transform the surd to eliminate the denominator under the radical:

(1) $\sqrt{\frac{2}{3}}$; (2) $\sqrt{1\frac{1}{7}}$;

(3) $\sqrt{\frac{4x}{3y}}$; (4) $\sqrt{\frac{a-5}{a+5}}$ ($a>5$).

Solution (1) $\sqrt{\frac{2}{3}}=\sqrt{\frac{2\times 3}{3\times 3}}=\frac{1}{3}\sqrt{6}$;

(2) $\sqrt{1\frac{1}{7}}=\sqrt{\frac{8}{7}}=\sqrt{\frac{8\times 7}{7\times 7}}=\sqrt{\frac{2^2\times 2\times 7}{7\times 7}}=\frac{2}{7}\sqrt{14}$;

(3) $\sqrt{\frac{4x}{3y}}=\sqrt{\frac{4x\bullet 3y}{3y\bullet 3y}}=\frac{2}{3y}\sqrt{3xy}$;

(4) $\sqrt{\frac{a-5}{a+5}}=\sqrt{\frac{(a-5)(a+5)}{(a+5)^2}}=\frac{1}{a+5}\sqrt{a^2-25}$.

Practice

1. Calculate:

(1) $\sqrt{\dfrac{25}{64}}$; (2) $\sqrt{\dfrac{4}{225}}$; (3) $\sqrt{\dfrac{0.01}{0.16}}$;

(4) $\sqrt{\dfrac{36\times 9}{121}}$; (5) $\sqrt{\dfrac{0.04\times 144}{0.49\times 169}}$; (6) $\sqrt{4\dfrac{1}{9}}$;

(7) $\sqrt{\dfrac{6}{4a^2}}$; (8) $\sqrt{\dfrac{49m^2n}{9c^2}}$.

2. Transform the surd by eliminating the denominator under the radical sign:

(1) $\sqrt{\dfrac{1}{2}}$; (2) $\sqrt{\dfrac{7}{12}}$; (3) $\sqrt{5\dfrac{1}{3}}$;

(4) $6\sqrt{\dfrac{5}{6}}$; (5) $\sqrt{\dfrac{27}{2x}}$; (6) $\sqrt{\dfrac{n}{3m^2}}$;

(7) $\sqrt{\dfrac{a}{50}}$; (8) $\sqrt{\dfrac{a-b}{a+b}}$ $(a>b)$.

3. (*Mental*) Is following equation correct? Why?

(1) $\sqrt{\dfrac{3}{4}}=2\sqrt{3}$; (2) $\sqrt{\dfrac{3}{2}}=\dfrac{1}{2}\sqrt{3}$;

(3) $\sqrt{\dfrac{8}{2}}=\sqrt{2}$; (4) $\sqrt{\dfrac{a}{9b}}=\dfrac{1}{3b}\sqrt{a}$.

10.3 Transforming Surd to Simplest Form and Identifying Like Surds

1. Surd in its Simplest Form

Let us look at the following example:

$$\sqrt{a^3b}=\sqrt{a^2\bullet ab}=\sqrt{a^2}\bullet\sqrt{ab}=a\sqrt{ab}$$

$$a^2\sqrt{\frac{b}{a}}=a^2\sqrt{\frac{ab}{a^2}}=\frac{a^2}{a}\sqrt{ab}=a\sqrt{ab}$$

Although the surds $\sqrt{a^3b}$ and $a^2\sqrt{\dfrac{b}{a}}$ are different in appearance, they can both be transformed into a comparatively simplified form as $a\sqrt{ab}$. When we compare $a\sqrt{ab}$ with $\sqrt{a^3b}$ and with $a^2\sqrt{\dfrac{b}{a}}$, we note that the surd in the form of $a\sqrt{ab}$ satisfies the following 2 conditions:

(1) The power of every factor under the radical sign is less than 2;

(2) There is no denominator under the radical sign.

A surd satisfying these 2 conditions is said to be expressed in its **simplest form**. For example: $4\sqrt{5a}$, $\dfrac{\sqrt{y}}{2}$, $\sqrt{a^2+b}$ are surds in simplest form, while $\sqrt{4a^3}$, $\sqrt{\dfrac{c}{3}}$ and $\sqrt{8}$ are not.

For a surd which is not presented in its simplest form, we can transform the surd to its simplest form using the method described above (i) to eliminate the denominator under the radical, and (i) to move the squared factors to the outside of the radical.

After transforming the surd to its simplest form, the surd will consist of two parts: (i) a part under the radical sign, and (ii) a part outside the radical sign which is the coefficient of the surd.

【Example】 Transform the following surd to its simplest form:

(1) $\sqrt{12}$; (2) $\sqrt{\dfrac{1}{3}}$;

(3) $4\sqrt{1\dfrac{1}{2}}$; (4) $x^2\sqrt{\dfrac{y}{x}}$.

Solution (1) $\sqrt{12}=\sqrt{2^2\times3}=2\sqrt{3}$;

(2) $\sqrt{\frac{1}{3}}=\sqrt{\frac{3}{3\times3}}=\frac{\sqrt{3}}{3}$;

(3) $4\sqrt{1\frac{1}{2}}=4\sqrt{\frac{3}{2}}=4\sqrt{\frac{6}{4}}=2\sqrt{6}$;

(4) $x^2\sqrt{\frac{y}{x}}=x^2\sqrt{\frac{xy}{x^2}}=x\sqrt{xy}$.

Note: When transforming a surd to its simplest form, an essential step is to factorise the values under the radical sign into prime factors.

Practice

1. Distinguish which surd is in its simplest form. If it is not, transform it to its simplest form.

(1) $\sqrt{45}$; (2) $\sqrt{25a^3}$; (3) $\sqrt{\frac{ab}{4}}$;

(4) $\sqrt{14}$; (5) $\sqrt{\frac{b}{a}}$; (6) $\frac{\sqrt{2}}{2}$;

(7) $\sqrt{6a^2b^3}$; (8) $\sqrt{\frac{4y}{5x}}$; (9) $\sqrt{a+b^2}$.

2. Transform the following surd into its simplest form:

(1) $3\sqrt{216}$; (2) $\sqrt{32}$; (3) $\sqrt{\frac{8}{9}}$;

(4) $\sqrt{1\frac{1}{3}}$; (5) $\sqrt{\frac{20a^2b}{a}}$; (6) $2\sqrt{a^3b^3}$;

(7) $x^2\sqrt{\frac{1}{8x^3}}$; (8) $\sqrt{\frac{ab}{(a+b)^2}}$; (9) $(a-b)\sqrt{\frac{1}{a-b}}$ $(a>b)$

2. Identifying Like Surds

By simplifying $\sqrt{12}$ and $\sqrt{\frac{1}{3}}$ to the simplest form, we have

$$\sqrt{12}=\sqrt{2^2\times3}=2\sqrt{3}$$

$$\sqrt{\frac{1}{3}}=\sqrt{\frac{3}{3\times3}}=\frac{1}{3}\sqrt{3}$$

The number under the radical sign for $2\sqrt{3}$ and $\frac{1}{3}\sqrt{3}$ are the same number, namely 3. If there are a number of surds, which after having been transformed to the simplest form, have the same number inside the radical, we call them **Like Surds**. For example, $\sqrt{12}$, $\sqrt{\frac{1}{3}}$ and $\frac{1}{2}\sqrt{3}$ are like surds. $a\sqrt{ab}$ and $3\sqrt{ab}$ are also like surds. However, $\sqrt{2}$ and $\sqrt{3}$ are unlike surds. Similarly, $\sqrt{a}$ and $\sqrt{3a}$ are also unlike surds.

We have learnt that, when adding like terms under a polynomiial, we can group and combine like terms together. In a similar manner, when adding like surds under an expression, we can group and combine like surds together.

【Example 1】 Idnetify from the following surds, which of them are like surds?

$$\sqrt{2},\ \sqrt{75},\ \sqrt{\frac{1}{50}},\ \sqrt{\frac{1}{27}},\ \sqrt{3},\ \frac{2}{3}\sqrt{8ab^3},\ 6b\sqrt{\frac{a}{2b}}.$$

Solution $\because$ $\sqrt{75}=\sqrt{5^2\times3}=5\sqrt{3}$

$$\sqrt{\frac{1}{50}}=\sqrt{\frac{2}{50\times2}}=\frac{1}{10}\sqrt{2}$$

$$\sqrt{\frac{1}{27}}=\sqrt{\frac{3}{27\times3}}=\frac{1}{9}\sqrt{3}$$

$$\frac{2}{3}\sqrt{8ab^3}=\frac{2}{3}\bullet 2b\sqrt{2ab}=\frac{4b}{3}\sqrt{2ab}$$

$$6b\sqrt{\frac{a}{2b}}=6b\sqrt{\frac{a\bullet 2b}{2b\bullet 2b}}=3\sqrt{2ab}$$

$\therefore\quad \sqrt{2}$, $\sqrt{\frac{1}{50}}$ are like surds.

$\sqrt{75}$, $\sqrt{\frac{1}{27}}$, $\sqrt{3}$ are like surds.

$\frac{2}{3}\sqrt{8ab^3}$, $6b\sqrt{\frac{a}{2b}}$ are like surds.

【**Example 2**】Group and combine like surds in the following expression

(1) $2\sqrt{2}-\frac{1}{2}\sqrt{3}+\frac{1}{3}\sqrt{2}-\sqrt{2}+\sqrt{3}$;

(2) $3\sqrt{xy}-a\sqrt{xy}+b\sqrt{xy}$ 。

Solution (1) $2\sqrt{2}-\frac{1}{2}\sqrt{3}+\frac{1}{3}\sqrt{2}-\sqrt{2}+\sqrt{3}$

$$=\left(2+\frac{1}{3}-1\right)\sqrt{2}+\left(-\frac{1}{2}+1\right)\sqrt{3}=\frac{4}{3}\sqrt{2}+\frac{1}{2}\sqrt{3}$$

(2) $3\sqrt{xy}-a\sqrt{xy}+b\sqrt{xy}=(3-a+b)\sqrt{xy}$

Practice

1. Identify which of the following surds are unlike surds?

(1) $\sqrt{63}$, $\sqrt{28}$;
(2) $\sqrt{12}$, $\sqrt{27}$, $4\sqrt{\frac{1}{3}}$;
(3) $\sqrt{4x^3}$, $2\sqrt{2x}$;
(4) $\sqrt{18}$, $\sqrt{50}$, $2\sqrt{\frac{2}{9}}$;
(5) $\sqrt{2x}$, $\sqrt{2a^2x^3}$, $\sqrt{50xy^2}$.

Practice

2. Group and combine like surds in the following expression:

(1) $6\sqrt{a}+2\sqrt{b}-4\sqrt{a}+3\sqrt{b}$;
(2) $\sqrt{5}+\sqrt{3}+2\sqrt{5}-\frac{\sqrt{3}}{3}-3\sqrt{5}$;
(3) $6\sqrt{3}+\sqrt{0.12}+\sqrt{48}$;
(4) $\frac{5}{2}\sqrt{xy}-2\sqrt{xy}-\frac{\sqrt{xy}}{2}$.

Exercise 3

1. For what value of real number a would the following expression be defined in the domain of real numbers?

$$\sqrt{a-2},\ \sqrt{2-a},\ \sqrt{a+2},\ \sqrt{(a-2)^2}$$

2. Rewrite the following expressions into difference of two squares, and further factorise it.

(1) x^2-9; (2) a^2-3; (3) $4a^2-7$; (4) $16b^2-11$.

3. Calculate:

(1) $(\sqrt{11})^2$; (2) $\sqrt{(-13)^2}$; (3) $-\sqrt{(5\times 6)^2}$;
(4) $\sqrt{a^6}$; (5) $\left(-7\sqrt{\frac{2}{7}}\right)^2$; (6) $\sqrt{(x+5)^2}$;
(7) $\sqrt{x^2-2x+1}$ ($x\geq 1$); (8) $(\sqrt{x-y})^2$ ($x\geq y$);
(9) $\sqrt{x^2-4x+4}$ ($x<2$);
(10) $x+y+\sqrt{x^2-2xy+y^2}$ ($x<y$).

4. Calculate:

(1) $\sqrt{9\times 25}$; (2) $\sqrt{36\times 256}$;
(3) $\sqrt{25\times 81\times 289}$; (4) $\sqrt{13^2-12^2}$;

(5) $\sqrt{65^2-16^2}$; (6) $\sqrt{9a^2}$;

(7) $\sqrt{(x+y)^2c^2}$; (8) $\sqrt{(a+b)^2(a-b)^2}$ ($a<b$).

5. Simplify:

(1) $\sqrt{5^6\times 3}$; (2) $\sqrt{242\times 49}$; (3) $\sqrt{(-32)(-15)}$;

(4) $\sqrt{4x^3}$; ; (5) $\sqrt{7a^4}$; (6) $\sqrt{5a(x+a)^3}$;

(7) $\sqrt{8(a+b)^4(c-d)^4}$; (8) $\sqrt{a^{2n}}$ (n is an integer).

6. Transferm the expression by moving the positive factors outside the radical sign to the inside of the radical sign without changing the value of the expression

(1) $2\sqrt{6}$; (2) $-5\sqrt{7}$; (3) $4\sqrt{\frac{1}{2}}$;

(4) $-2a\sqrt{b}$; (5) $\frac{2}{3}\sqrt{3}$; (6) $ab\sqrt{\frac{1}{a}+\frac{1}{b}}$.

7. Calculate:

(1) $\sqrt{\frac{9}{49}}$; (2) $\sqrt{2\frac{34}{81}}$; (3) $\sqrt{\frac{0.16}{0.0225}}$;

(4) $\sqrt{\frac{0.01\times 64}{0.36\times 4}}$; (5) $\sqrt{\frac{27}{100}}$; (6) $\sqrt{\frac{25y^4}{121x^6}}$;

(7) $\sqrt{\frac{18a^2}{4b^2}}$; (8) $\sqrt{\left(1\frac{1}{25}\right)^2-\left(\frac{2}{5}\right)^2}$ (9) $\sqrt{130^2-66^2}$.

8. Transform the surd by eliminating the denominator under the rdical sign:

(1) $\sqrt{\frac{1}{6}}$; (2) $\sqrt{2\frac{3}{11}}$; (3) $\sqrt{\frac{127}{32}}$;

(4) $8\sqrt{\frac{3}{128}}$; (5) $\sqrt{\frac{n^3}{9m}}$; (6) $a\sqrt{\frac{b}{a^5}}$;

(7) $\sqrt{\frac{7(a-b)}{27(a+b)}}$ ($a>b$); (8) $a\sqrt{\frac{1}{a^2}-\frac{1}{b^2}}$ ($a<b$);

(9) $\sqrt{\frac{a+b}{(a-b)^2}}$ ($a<b$).

9. Transform the surd to the simplest form:

(1) $\sqrt{72}$; (2) $6\sqrt{\frac{1}{8}}$; (3) $10\sqrt{1\frac{4}{5}}$;

(4) $\sqrt{(-8)^2-4\times(-4)}$ (5) $\sqrt{\left(3\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}$ (6) $\sqrt{\frac{x}{y}}$;

(7) $\sqrt{25m^3+50m^2}$; (8) $\frac{2a^2}{3b}\sqrt{\frac{b^3}{a^4}-\frac{b^2}{a^4}}$ ($b>1$);

(9) $\frac{a}{a-2b}\sqrt{\frac{a^2b-4ab^2+4b^3}{a}}$ ($a<2b$).

10. When $a=1$, $b=10$, $c=-15$, find the value of the expression $\frac{-b+\sqrt{b^2-4ac}}{2a}$ (express the surd in the simplest form in the answer).

11. When $a=2$, $b=-8$, $c=5$, find the value of the expression $\frac{-b-\sqrt{b^2-4ac}}{2a}$ (express the surd in the simplest form in the answer).

12. In the following surds, which of them are like surds?

$\sqrt{8}$, $\sqrt{20}$, $-\sqrt{\frac{5}{16}}$, $\sqrt{\frac{1}{18}}$, $3\sqrt{\frac{4}{5}}$, $-\sqrt{121a^3}$, $a\sqrt{\frac{1}{a}}$, $2\sqrt{a^3b^3c}$,

$3\sqrt{a^3bc^3}$, $4\sqrt{\frac{c}{ab}}$, $\sqrt{\frac{1}{mn-np}}$ ($m>p$), $\sqrt{\frac{n^3}{m-p}}$ ($m>p$).

13. Group and combine the like surds in the following expression.

(1) $\sqrt{2}+\sqrt{3}+3\sqrt{2}-\dfrac{\sqrt{3}}{3}-\dfrac{\sqrt{2}}{2}$;

(2) $\sqrt{125}+3\sqrt{\dfrac{2}{27}}-4\sqrt{216}-3\sqrt{\dfrac{1}{5}}$;

(3) $5\sqrt{xy}-7\sqrt{x}-3\sqrt{yx}+4\sqrt{x}$;

(4) $2a\sqrt{3ab^2}-\left(\dfrac{b}{5}\sqrt{27a^3}-2ab\sqrt{\dfrac{3a}{4}}\right)$.

10.4 Addition and subtraction of Surds

Addition and subtraction of Surds is the grouping and combining of like surds, similar to the addition and subtraction of like terms in polynomials. Before we can combine like surds, we shall need to transform all surds to the simplest form first. In other words, to perform addition and subtraction of expressions involving surds, we shall need to transform all surds into the simplest form, then we can group and combine like surds together.

【Example 1】 Calculate $2\sqrt{12}-4\sqrt{\dfrac{1}{27}}+3\sqrt{48}$.

Solution $2\sqrt{12}-4\sqrt{\dfrac{1}{27}}+3\sqrt{48}=4\sqrt{3}-\dfrac{4}{9}\sqrt{3}+12\sqrt{3}$

$$=\left(4-\frac{4}{9}+12\right)\sqrt{3}$$

$$=\frac{140}{9}\sqrt{3}$$

【Example 2】 Calculate $\dfrac{2}{3}\sqrt{9x}+6\sqrt{\dfrac{x}{4}}-2x\sqrt{\dfrac{1}{x}}$.

Solution $\dfrac{2}{3}\sqrt{9x}+6\sqrt{\dfrac{x}{4}}-2x\sqrt{\dfrac{1}{x}}=2\sqrt{x}+3\sqrt{x}-2\sqrt{x}=3\sqrt{x}$.

【Example 3】 Calculate $\left(\sqrt{32}+\sqrt{0.5}-2\sqrt{\dfrac{1}{3}}\right)-\left(\sqrt{\dfrac{1}{8}}-\sqrt{75}\right)$.

Solution $\left(\sqrt{32}+\sqrt{0.5}-2\sqrt{\dfrac{1}{3}}\right)-\left(\sqrt{\dfrac{1}{8}}-\sqrt{75}\right)$

$$=\sqrt{32}+\sqrt{\frac{1}{2}}-2\sqrt{\frac{1}{3}}-\sqrt{\frac{1}{8}}-\sqrt{75}$$

$$=4\sqrt{2}+\frac{1}{2}\sqrt{2}-\frac{2}{3}\sqrt{3}-\frac{1}{4}\sqrt{2}+5\sqrt{3}$$

$$=\left(4+\frac{1}{2}-\frac{1}{4}\right)\sqrt{2}+\left(5-\frac{2}{3}\right)\sqrt{3}$$

$$=\frac{17}{4}\sqrt{2}+\frac{13}{3}\sqrt{3}$$

Practice

1. Calculate:

(1) $5\sqrt{2}+\sqrt{8}-7\sqrt{18}$;

(2) $\sqrt{28}+9\sqrt{112}$;

(3) $3\sqrt{40}-\sqrt{\dfrac{2}{5}}-2\sqrt{\dfrac{1}{10}}$;

(4) $\sqrt{12}+\sqrt{\dfrac{1}{27}}-\sqrt{\dfrac{1}{3}}$;

(5) $\dfrac{1}{3}\sqrt{32}+\dfrac{\sqrt{8}}{2}-\dfrac{1}{5}\sqrt{50}$;

(6) $\sqrt{2x}-\sqrt{8x^3}+2\sqrt{2xy^2}$;

(7) $x\sqrt{\dfrac{1}{x}}+\sqrt{4y}-\dfrac{\sqrt{x}}{2}+y\sqrt{\dfrac{1}{y}}$;

(8) $\dfrac{-b+\sqrt{b^2-4ac}}{2a}+\dfrac{-b-\sqrt{b^2-4ac}}{2a}$ $(b^2>4ac)$.

Practice

2. Calculate:

(1) $\sqrt{18}-(\sqrt{98}-2\sqrt{75}+\sqrt{27})$;

(2) $(\sqrt{45}+\sqrt{108})+\left(\sqrt{1\frac{1}{3}}-\sqrt{125}\right)$;

(3) $\left(\sqrt{24}-\sqrt{0.5}-2\sqrt{\frac{2}{3}}\right)-\left(\sqrt{\frac{1}{8}}-\sqrt{6}\right)$;

(4) $\frac{1}{2}(\sqrt{2}+\sqrt{3})-\frac{3}{4}(\sqrt{2}-\sqrt{27})$;

(5) $a^2\sqrt{8a}+3a\sqrt{50a^3}-\frac{a}{2}\sqrt{18a^3}$;

(6) $\left(4b\sqrt{\frac{a}{b}}+\frac{2}{a}\sqrt{a^3b}\right)-\left(3a\sqrt{\frac{b}{a}}+\sqrt{9ab}\right)$.

3. Find the value of the following expression (correct to the nearest 0.01):

(1) $\sqrt{2\frac{2}{3}}+\sqrt{\frac{2}{3}}-\frac{1}{5}\sqrt{54}$;

(2) $\left(5\sqrt{\frac{1}{5}}+\frac{1}{2}\sqrt{20}\right)-\left(\frac{5}{4}\sqrt{\frac{4}{5}}-\sqrt{45}\right)$.

4. Is the following equation correct? Why?

(1) $\sqrt{2}+\sqrt{3}=\sqrt{5}$;

(2) $2+\sqrt{2}=2\sqrt{2}$;

(3) $a\sqrt{x}-b\sqrt{x}=(a-b)\sqrt{x}$;

(4) $\frac{\sqrt{8}+\sqrt{18}}{2}=\sqrt{4}+\sqrt{9}=2+3=5$.

10.5 Multiplication of Surds

By reversing the equation $\sqrt{ab}=\sqrt{a}\bullet\sqrt{b}$, we get

$$\boldsymbol{\sqrt{a}\bullet\sqrt{b}=\sqrt{ab}}.$$

We can apply this formula to calculate the Multiplication of Surds. From the formula, we know that the product of two surds is a surd with the value under the radical sign being the product of the values under the radical sign of the original surds.

【Example 1】 Calculate: (1) $\sqrt{14}\bullet\sqrt{7}$; (2) $3\sqrt{5a}\bullet 2\sqrt{10b}$.

Solution (1) $\sqrt{14}\bullet\sqrt{7}=\sqrt{14\times 7}=\sqrt{7^2\times 2}=7\sqrt{2}$;

(2) $3\sqrt{5a}\bullet 2\sqrt{10b}=3\times 2\sqrt{5a\bullet 10b}=30\sqrt{2ab}$.

Note: If the result of an operation on surds gives rise to some surds, in general, we shall need to transform the surds to the simplest form to see if they can be further simplified.

【Example 2】 Calculate: (1) $\left(\sqrt{\frac{8}{27}}-5\sqrt{3}\right)\bullet\sqrt{6}$;

(2) $(5+\sqrt{6})(5\sqrt{2}-2\sqrt{3})$.

Solution (1)
$$\begin{aligned}\left(\sqrt{\frac{8}{27}}-5\sqrt{3}\right)\bullet\sqrt{6}&=\sqrt{\frac{8}{27}}\bullet\sqrt{6}-5\sqrt{3}\bullet\sqrt{6}\\&=\sqrt{\frac{8}{27}\times 6}-5\sqrt{3\times 6}\\&=\frac{4}{3}-15\sqrt{2}\end{aligned}$$

(2)
$$\begin{aligned}(5+\sqrt{6})(5\sqrt{2}-2\sqrt{3})&=25\sqrt{2}-10\sqrt{3}+5\sqrt{12}-2\sqrt{18}\\&=25\sqrt{2}-10\sqrt{3}+10\sqrt{3}-6\sqrt{2}\\&=19\sqrt{2}\end{aligned}$$

Multiplication of a sum of surds by another sum of surds is similar to the multiplication of polynomials. Whatever multiplicative rules that can be applied to the multiplication of polynomials can also be applied to the multiplication of the sum of surds.

【Example 3】 Calculate: (1) $(2\sqrt{3}+3\sqrt{2})(2\sqrt{3}-3\sqrt{2})$;

(2) $(4+3\sqrt{5})^2$;

(3) $(\sqrt{6}-3\sqrt{3})^2$.

Solution (1)
$$(2\sqrt{3}+3\sqrt{2})(2\sqrt{3}-3\sqrt{2})=(2\sqrt{3})^2-(3\sqrt{2})^2 =12-18 =-6$$

(2)
$$(4+3\sqrt{5})^2=4^2+2\cdot 4\cdot 3\sqrt{5}+(3\sqrt{5})^2 =16+24\sqrt{5}+45 =61+24\sqrt{5}$$

(3)
$$(\sqrt{6}-3\sqrt{3})^2=(\sqrt{6})^2-2\cdot\sqrt{6}\cdot 3\sqrt{3}+(3\sqrt{3})^2 =6-18\sqrt{2}+27 =33-18\sqrt{2}$$

【Example 4】 Calculate: (1) $(\sqrt{3}+\sqrt{6})(\sqrt{3}-\sqrt{6})$;

(2) $(2\sqrt{ax}-5\sqrt{by})(2\sqrt{ax}+5\sqrt{by})$.

Solution (1) $(\sqrt{3}+\sqrt{6})(\sqrt{3}-\sqrt{6})=(\sqrt{3})^2-(\sqrt{6})^2=3-6=-3$;

(2)
$$(2\sqrt{ax}-5\sqrt{by})(2\sqrt{ax}+5\sqrt{by})=(2\sqrt{ax})^2-(5\sqrt{by})^2 =4ax-25by$$

Practice

1. Calculate:

(1) $\sqrt{5}\cdot\sqrt{3}$; (2) $6\sqrt{27}\cdot(-2\sqrt{3})$;

(3) $9\sqrt{45}\times\frac{3}{2}\sqrt{2\frac{2}{3}}$; (4) $\sqrt{6x}\cdot\sqrt{2x}$;

(5) $\frac{a}{b}\sqrt{\frac{b}{a}}\cdot\frac{b}{a}\sqrt{\frac{a}{b}}$; (6) $10x\sqrt{y}\cdot\sqrt{\frac{1}{x}}$.

Practice

2. Calculate:

(1) $(\sqrt{12}-3\sqrt{75})\cdot\sqrt{3}$; (2) $2\sqrt{5}(\sqrt{10}+4\sqrt{12})$;

(3) $(\sqrt{2}+2\sqrt{12}-\sqrt{6})\cdot 2\sqrt{3}$; (4) $3\sqrt{6}(3\sqrt{2}-\sqrt{15})$.

3. Calculate:

(1) $(2\sqrt{3}-2)(3\sqrt{2}-3)$;

(2) $\left(\frac{\sqrt{5}}{3}-2\sqrt{3}\right)\left(3\sqrt{5}-\frac{1}{2}\sqrt{3}\right)$;

(3) $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{c})$; (4) $(2\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$.

4. Calculate:

(1) $(4-3\sqrt{5})(4+3\sqrt{5})$; (2) $(7\sqrt{2}+2\sqrt{6})(2\sqrt{6}-7\sqrt{2})$;

(3) $(\sqrt{4x+3}-\sqrt{2x})(\sqrt{4x+3}+\sqrt{2x})$;

(4) $(\sqrt{3}+2\sqrt{2})^2$; (5) $\left(\frac{-1-\sqrt{3}}{2}\right)^2$;

(6) $(4\sqrt{7}-7\sqrt{3})^2$; (7) $\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)^2$;

(8) $(\sqrt{x}+\sqrt{y})^2+(\sqrt{x}-\sqrt{y})^2$;

(9) $(\sqrt{2}+\sqrt{3}-\sqrt{6})^2-(\sqrt{2}-\sqrt{3}+\sqrt{6})^2$;

(10) $(1+\sqrt{2}-\sqrt{3})(1-\sqrt{2}+\sqrt{3})$.

10.6 Division of Surds

By reversing the equation $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$, we get

$$\boldsymbol{\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}}.$$

We can apply this formula to perform the operation of division of surds. For example, if we need to find the quotient of $\sqrt{a} \div \sqrt{b}$, we can apply the formula to find the square root of the quotient $a \div b$, which is $\sqrt{\frac{a}{b}}$.

【Example 1】 Calculate: (1) $\sqrt{72} \div \sqrt{6}$;

(2) $\sqrt{1\frac{1}{2}} \div \sqrt{\frac{1}{6}}$.

Solution (1) $\sqrt{72} \div \sqrt{6} = \frac{\sqrt{72}}{\sqrt{6}} = \sqrt{\frac{72}{6}} = \sqrt{12} = 2\sqrt{3}$;

(2) $\sqrt{1\frac{1}{2}} \div \sqrt{\frac{1}{6}} = \frac{\sqrt{\frac{3}{2}}}{\sqrt{\frac{1}{6}}} = \sqrt{\frac{3}{2} \times 6} = \sqrt{9} = 3$.

When the surd is in the form of $\sqrt{\frac{a}{b}}$, we can apply fraction reduction process to reduce the fraction by cancelling common factors between *a* and *b*. If the denominator cannot be totally removed after the fraction reduction process, we shall change the surd back to the form $\frac{\sqrt{a}}{\sqrt{b}}$ and apply another process to eliminate the surd in the denominator, .which is to multiply both the numerator and denominator by $\sqrt{b}$.

For example, to remove the surd in the denominator of $\frac{\sqrt{3}}{\sqrt{2}}$, we multiply both numerator and denominator by $\sqrt{2}$ as follows:

$$\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3} \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{6}}{(\sqrt{2})^2} = \frac{1}{2}\sqrt{6}.$$

In a similar manner, to simplify $\frac{1}{\sqrt{3} - \sqrt{2}}$, we multiply both the numerator and denominator by the same factor $(\sqrt{3} + \sqrt{2})$, as follows:

$$\frac{1}{\sqrt{3} - \sqrt{2}} = \frac{\sqrt{3} + \sqrt{2}}{(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})}$$
$$= \frac{\sqrt{3} + \sqrt{2}}{3 - 2}$$
$$= \sqrt{3} + \sqrt{2}$$

The process of eliminating the surd in the denominator is called **Rationalising the Denominator** (or Denominator Rationisation).

If two expressions with surds are multiplied together, the resulting expression does not contain any surd, we say that **the two expressions are rationalising factors to each other**.

In the above example, $\sqrt{2}$ and $\sqrt{2}$, $\sqrt{3} + \sqrt{2}$ and $\sqrt{3} - \sqrt{2}$ are rationalisation factors to each other or mutually rationalisation factors.

【Example 2】 Rationalize the denominator of the following expression:

(1) $\frac{1}{\sqrt{5}}$; (2) $\frac{4}{3\sqrt{7}}$;

(3) $\frac{a}{\sqrt{a+b}}$; (4) $\frac{\sqrt{5a}}{\sqrt{20a}}$.

Solution (1) $\frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5} \bullet \sqrt{5}} = \frac{\sqrt{5}}{5}$;

(2) $\frac{4}{3\sqrt{7}} = \frac{4 \bullet \sqrt{7}}{3\sqrt{7} \bullet \sqrt{7}} = \frac{4}{21}\sqrt{7}$;

(3) $\frac{a}{\sqrt{a+b}} = \frac{a \bullet \sqrt{a+b}}{\sqrt{a+b} \bullet \sqrt{a+b}} = \frac{a}{a+b}\sqrt{a+b}$;

(4) $\dfrac{\sqrt{5a}}{\sqrt{20a}}=\dfrac{\sqrt{5}(\sqrt{a})^2}{2\sqrt{5}\bullet\sqrt{a}}=\dfrac{1}{2}\sqrt{a}$.

It can be observed from Questions (1) to (3) of Example 2 that the rationalising factor of $\sqrt{a}$ is $\sqrt{a}$. Sometimes, in the fraction reduction process, the denominator is totally removed, and there is no need to perform the rationalisation process. This can be seen in Question (4) of Example 2.

【Example 3】 Rationalise the denominator of the following expression:

(1) $\dfrac{1}{\sqrt{2}+1}$; (2) $\dfrac{\sqrt{2}}{3-\sqrt{3}}$;

(3) $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ $(x\neq y)$; (4) $\dfrac{x-y}{\sqrt{x}+\sqrt{y}}$.

Solution (1) $\dfrac{1}{\sqrt{2}+1}=\dfrac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)}=\dfrac{\sqrt{2}-1}{2-1}=\sqrt{2}-1$;

(2) $\dfrac{\sqrt{2}}{3-\sqrt{3}}=\dfrac{\sqrt{2}(3+\sqrt{3})}{(3-\sqrt{3})(3+\sqrt{3})}=\dfrac{3\sqrt{2}+\sqrt{6}}{9-3}=\dfrac{3\sqrt{2}+\sqrt{6}}{6}$;

(3) $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}=\dfrac{(\sqrt{x}-\sqrt{y})^2}{(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}=\dfrac{x+y-2\sqrt{xy}}{x-y}$;

(4) $$\dfrac{x-y}{\sqrt{x}+\sqrt{y}}=\dfrac{(\sqrt{x})^2-(\sqrt{y})^2}{\sqrt{x}+\sqrt{y}}=\dfrac{(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}{\sqrt{x}+\sqrt{y}}$$
$$=\sqrt{x}-\sqrt{y}$$

It can be observed from Questions (1) to (3) in the above example that $a\sqrt{x}+b\sqrt{y}$ and $a\sqrt{x}-b\sqrt{y}$ are rationalising factors of each other. Further, it can be observed from Question (4) that, when a fraction is involved, it is usually simpler to apply fraction

reduction process first to reduce the fraction by cancelling common factors. If the denominator is not totally removed, we shall then apply the denominator rationalisation process to eliminate the denominator.

【Example 4】 Calculate: (1) $(6\sqrt{7}-4\sqrt{2})\div\sqrt{3}$;

(2) $(\sqrt{12}-5\sqrt{8})\div(\sqrt{6}+\sqrt{2})$.

Solution (1) $$(6\sqrt{7}-4\sqrt{2})\div\sqrt{3}=\dfrac{(6\sqrt{7}-4\sqrt{2})\bullet\sqrt{3}}{\sqrt{3}\bullet\sqrt{3}}$$
$$=2\sqrt{21}-\dfrac{4}{3}\sqrt{6}$$

(2) $$(\sqrt{12}-5\sqrt{8})\div(\sqrt{6}+\sqrt{2})=\dfrac{\sqrt{12}-5\sqrt{8}}{\sqrt{6}+\sqrt{2}}$$
$$=\dfrac{(2\sqrt{3}-10\sqrt{2})(\sqrt{6}-\sqrt{2})}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}$$
$$=\dfrac{6\sqrt{2}-2\sqrt{6}-20\sqrt{3}+20}{4}$$
$$=\dfrac{3}{2}\sqrt{2}-\dfrac{1}{2}\sqrt{6}-5\sqrt{3}+5$$

In general, regarding the division of Surds, we can first write it in fractional form, followed by denominator rationalisation.

Practice

1. Calculate:

(1) $-\sqrt{54}\div\sqrt{3}$; (2) $\sqrt{1\dfrac{3}{5}}\div\sqrt{3\dfrac{1}{5}}$;

(3) $6\sqrt{3}\div3\sqrt{6}$; (4) $\dfrac{1}{2}\sqrt{6}\bullet4\sqrt{\dfrac{1}{12}}\div\dfrac{2}{3}\sqrt{1\dfrac{1}{2}}$;

(5) $4\sqrt{6a^3}\div2\sqrt{\dfrac{a}{3}}$; (6) $a^2x\div\sqrt{ax^3}$.

Practice

2. Rationalise the denominator of the following expression:

(1) $\frac{1}{\sqrt{3}}$; (2) $\frac{\sqrt{3}}{\sqrt{40}}$; (3) $\frac{x^2}{\sqrt{4xy}}$;

(4) $\frac{2n}{3\sqrt{n}}$; (5) $\frac{a^2-b^2}{\sqrt{a+b}}$.

3. Calculate:

(1) $\sqrt{3}\div(5-\sqrt{7})-(2\sqrt{3}-5\sqrt{7})^2$;

(2) $(2\sqrt{3}-2)(3\sqrt{6}+\sqrt{2})+(3\sqrt{3}+5\sqrt{2})\div(3\sqrt{3}-5\sqrt{2})$.

Exercise 4

1. Calculate:

(1) $3\sqrt{8}+2\sqrt{32}-\sqrt{50}$; (2) $9\sqrt{3}-7\sqrt{12}+5\sqrt{48}$;

(3) $6-\sqrt{\frac{3}{2}}-\sqrt{\frac{2}{3}}$; (4) $2\sqrt{\frac{2}{3}}+\sqrt{\frac{1}{6}}-\frac{1}{5}\sqrt{54}$;

(5) $\sqrt{\frac{1}{5}}+2\sqrt{20}-4\sqrt{\frac{1}{5}}-\frac{1}{5}\sqrt{5}$;

(6) $\sqrt{12}+3\sqrt{1\frac{1}{3}}-\sqrt{5\frac{1}{3}}-\frac{2}{3}\sqrt{48}$;

(7) $2a\sqrt{27a}+6a\sqrt{\frac{3a}{4}}$;

(8) $\frac{1}{a}\sqrt{x^3}-\sqrt{\frac{1}{x}}$;

(9) $2a\sqrt{3ab^2}-\frac{b}{6}\sqrt{27a^3}+2ab\sqrt{\frac{3}{4}a}$;

(10) $5\sqrt{x^3y}-2y\sqrt{xy}-6\sqrt{xy}+\sqrt{\frac{y}{x}}+\sqrt{\frac{x}{y}}$.

2. Calculate:

(1) $(\sqrt{18}-\sqrt{98})+(2\sqrt{75}-\sqrt{27})$;

(2) $(\sqrt{45}+\sqrt{18})-(\sqrt{8}-\sqrt{125})$;

(3) $\left(\sqrt{12}-\sqrt{\frac{1}{2}}-2\sqrt{\frac{1}{3}}\right)-2\left(\sqrt{\frac{1}{8}}-\sqrt{18}\right)$;

(4) $\left(5\sqrt{\frac{1}{5}}-\frac{1}{2}\sqrt{20}\right)+\left(\frac{5}{4}\sqrt{\frac{4}{5}}-\sqrt{45}\right)$;

(5) $7\sqrt{a}-\left(a\sqrt{\frac{1}{a}}-4\sqrt{ab^2}\right)$;

(6) $\left(\frac{2}{3}x\sqrt{9x}+6x\sqrt{\frac{y}{x}}\right)+\left(\sqrt{\frac{x}{y}}-x^2\sqrt{\frac{1}{x}}\right)$.

3. (1) When $x=7$, calculate the value of the following expression:

$$\sqrt{x+5}+\sqrt{x-4}-\sqrt{4x-1}.$$

(2) When $x=4$ and $y=16$, calculate the value of the following expression:

$$\sqrt{x^3+x^2y+\frac{1}{4}xy^2}+\sqrt{\frac{1}{4}x^2y+xy^2+y^3}$$

4. Calculate:

(1) $\sqrt{15}\bullet\sqrt{1\frac{2}{3}}$; (2) $6\sqrt{1\frac{3}{5}}\bullet\left(-5\sqrt{2\frac{2}{5}}\right)$;

(3) $\frac{1}{2}\sqrt{12x}\bullet 2\sqrt{3x}$;

(4) $10a^2\sqrt{ab}\bullet 5\sqrt{\frac{1}{a}}$;

(5) $\frac{1}{3}\sqrt{30}\bullet 40\sqrt{\frac{1}{2}}\bullet\frac{3}{2}\sqrt{2\frac{2}{3}}$;

(6) $3\sqrt{\frac{a}{x}}\bullet\left(-2\sqrt{\frac{x}{a}}\right)\bullet\sqrt{\frac{b}{a}}$.

5. Calculate:

(1) $(\sqrt{12}+5\sqrt{8})\bullet\sqrt{3}$;

(2) $3\sqrt{2}\bullet\left(2\sqrt{12}-4\sqrt{\frac{1}{8}}+3\sqrt{48}\right)$;

(3) $\left(\sqrt{xy}-2\sqrt{\frac{y}{x}}+\sqrt{\frac{x}{y}}\right)\bullet\sqrt{xy}$;

(4) $(\sqrt{a^3b}+\sqrt{ab^3}-ab)\bullet\sqrt{ab}$;

(5) $\frac{3}{4}\left(\frac{\sqrt{5}}{3}-2\sqrt{3}\right)-\frac{\sqrt{3}}{2}(1-4\sqrt{3}+3\sqrt{5})$.

6. Calculate:

(1) $(2\sqrt{3}-2)(3\sqrt{6}+\sqrt{2})$;

(2) $(\sqrt{27}+\sqrt{28})(\sqrt{12}-\sqrt{63})$;

(3) $(2\sqrt{3}-3\sqrt{2}+\sqrt{6})(\sqrt{6}-5\sqrt{3})$;

(4) $(\sqrt{5}+\sqrt{3}+\sqrt{2})(\sqrt{5}-2\sqrt{3}+\sqrt{2})$;

(5) $(\sqrt{x}+\sqrt{3})(2\sqrt{x}+3\sqrt{2})$;

(6) $(x+y+2\sqrt{xy})(\sqrt{x}-\sqrt{y})$.

7. Calculate:

(1) $(5\sqrt{3}+4\sqrt{2})(5\sqrt{3}-4\sqrt{2})$;

(2) $(7\sqrt{5}+6\sqrt{7})(6\sqrt{7}-7\sqrt{5})$;

(3) $(3\sqrt{2}+\sqrt{48})(\sqrt{18}-4\sqrt{3})$;

(4) $\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$ $(b^2-4ac>0)$;

(5) $(7\sqrt{3}+2\sqrt{7})^2$;

(6) $(4-5\sqrt{3})^2$;

(7) $\left(3\sqrt{a}+2\sqrt{\frac{x}{a}}\right)^2$;

(8) $\left(3\sqrt{1\frac{2}{3}}-\sqrt{1\frac{4}{5}}\right)^2$;

(9) $(\sqrt{2}+\sqrt{3}-\sqrt{6})(\sqrt{2}-\sqrt{3}-\sqrt{6})$;

(10) $(\sqrt{x+y}+\sqrt{x-y})^2+(\sqrt{x+y}-\sqrt{x-y})^2$ $(x>y)$.

8. Rationalise the denominator of the following expression:

(1) $\frac{3}{\sqrt{5}}$; (2) $\frac{\sqrt{2}}{3\sqrt{40}}$; (3) $\frac{7n}{3\sqrt{n}}$;

(4) $\frac{\sqrt{x-1}}{\sqrt{x+1}}$ $(x>1)$; (5) $\frac{1}{\sqrt{5}-2}$;

(6) $\frac{\sqrt{5}}{\sqrt{3}+\sqrt{2}}$; (7) $\frac{\sqrt{3}+\sqrt{15}}{\sqrt{3}-\sqrt{15}}$;

(8) $\frac{3\sqrt{5}-2\sqrt{3}}{3\sqrt{5}+2\sqrt{3}}$; (9) $\frac{2\sqrt{x+2}+3\sqrt{x-2}}{\sqrt{x+2}+\sqrt{x-2}}$ $(x>2)$.

9. Calculate:

(1) $\sqrt{\frac{1}{45}}\div\frac{3}{2}\sqrt{2\frac{2}{3}}$; (2) $\sqrt{20a}\div\frac{2}{3}\sqrt{b}$;

(3) $\sqrt{15}\bullet\sqrt{1\frac{2}{3}}\div\sqrt{24}$;

(4) $\sqrt{\frac{a}{b}}\bullet\left(\sqrt{\frac{b}{a}}\div\sqrt{\frac{1}{b}}\right)$ $(x>1)$;

(5) $\left(\sqrt{48}+\frac{1}{2}\sqrt{1\frac{1}{2}}\right)\div\sqrt{27}$;

(6) $\left(\frac{3}{4}\sqrt{7}-\sqrt{6}\right)\div(\sqrt{5}-\sqrt{3})$;

(7) $(7\sqrt{2}+2\sqrt{6})\div(2\sqrt{6}-7\sqrt{2})$;

(8) $\sqrt{15}\div\left(\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{2}}\right)$;

10. Calculate the value of the following expression (correct to the nearest 0.01):

(1) $\dfrac{\sqrt{5}-1}{\sqrt{2}}$; (2) $\dfrac{\sqrt{2}}{3-\sqrt{3}}$;

(3) $\dfrac{3}{2\sqrt{2}}-\dfrac{2}{\sqrt{5}+1}$; (4) $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$;

(5) $\left(\dfrac{2}{x+1}\right)^2$, where $x=\sqrt{3}$;

(6) $\dfrac{a-4b}{\sqrt{a}-2\sqrt{b}}$, where $a=6$, $b=5$.

11. (1) Given $x=\dfrac{2}{\sqrt{3}-1}$, find the value of x^2-x+1;

(2) Given $x=2+\sqrt{3}$, find the value of $\dfrac{3x^2-2x+5}{2x-7}$.

12. Simplify:

(1) $\dfrac{1}{\sqrt{3}+\sqrt{2}}+\dfrac{1}{\sqrt{2}+1}-\dfrac{2}{\sqrt{3}+1}$;

(2) $\dfrac{5}{4-\sqrt{11}}+\dfrac{1}{3+\sqrt{7}}-\dfrac{6}{\sqrt{7}-2}-\dfrac{\sqrt{7}-5}{2}$;

(3) $\dfrac{1}{y}\sqrt{x-y}+\dfrac{x}{\sqrt{x-y}}-\dfrac{1}{x-y}\sqrt{(x-y)^3}$ ($x>y$);

(4) $\dfrac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}+\dfrac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}-\sqrt{x}}$.

Chapter Summary

I. This chapter mainly teaches the basic property and operation of Surds (also known as Radicals).

II. Basing on the property of principal square root being unique, we have derived the following basic properties for Surds:

$$(\sqrt{a})^2=a \qquad (a\ge 0)$$

$$\sqrt{a^2}=|a|=\begin{cases} a & (a>0)\\ 0 & (a=0)\\ -a & (a<0)\end{cases}$$

$$\sqrt{ab}=\sqrt{a}\bullet\sqrt{b} \qquad (a\ge 0、b\ge 0)$$

$$\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}} \qquad (a\ge 0、b>0)$$

III. From the basic property of Surds, we have derived the operation rules and simplification process for Surds.

A Surd in its simplest form satisfies the following 2 conditions:

(1) The power of every factor unde the radical sign is less than 2;

(2) There is no denominator under the radical sign.

When Surds are transformed into the simplest form, if two surds have the same values under the radical sign, we call them Like Surds, and can combine them together into one surd.

For additions and subtractions, it is essential that we transform all surds to the simplest form first. Then if there are Like Surds, we can group them and combine them in the same way that we groupd and combine like terms in a polynomial.

For multiplication of Surds, we operate using the formula $\sqrt{a}\bullet\sqrt{b}=\sqrt{ab}$ ($a\ge 0$, $b\ge 0$) and the multiplication rules for polynomials.

For division of Surds, we operate using the formula $\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$ ($a \geq 0$, $b>0$). For the fraction $\frac{a}{b}$ under the radical sign, we first apply fraction reduction process to reduce the fraction to its simplest form to see if the denominator can be totally removed. If the denominator cannot be totally removed, we shall transform the radical back to the form $\frac{\sqrt{a}}{\sqrt{b}}$ and simplify the surd by applying the denominator rationalisation process to eliminate the denominator.

It is customary practice that, at the end of the operation, all surds that result from the operation are transformed to the simplest form.

Revision Exercise 10

1. For what value of x would the following expression be defined in the domain of real numbers?

   (1) $\sqrt{x-3}$; (2) $\sqrt{3-x}$; (3) $\sqrt{1+x^2}$;

   (4) $\sqrt{\frac{1}{x^2}}$; (5) $\sqrt{x}+\sqrt{-x}$; (6) $\frac{1}{1-\sqrt{x}}$.

2. Factorise the following expression in the domain of real numbers.

   (1) x^2-7; (2) $4a^4-1$; (3) a^4-6a^2+9;

   (4) $m^4-10m^2n^2+25n^4$.

3. Explain how the incorrect result is derived wrongly from the steps::

   (1) $\because$ $(-3)^2=3^2$

   $\therefore$ $\sqrt{(-3)^2}=\sqrt{3^2}$

   $\because$ $\sqrt{(-3)^2}=-3$ and $\sqrt{3^2}=3$

   $\therefore$ $-3=3$

   (2) $\because$ $-2\sqrt{3}=\sqrt{(-2)^2\times 3}=\sqrt{12}$ and $\sqrt{12}=2\sqrt{3}$

   $\therefore$ $-2\sqrt{3}=2\sqrt{3}$

   $\therefore$ $-2=2$

4. What is meant by a surd in the simplest form? Please transform the following surd to the simplest form:

   (1) $\sqrt{500}$; (2) $\sqrt{4\frac{2}{3}}$; (3) $\sqrt{12x}$;

   (4) $\sqrt{3a^2b^2}$ ($b<0$); (5) $\sqrt{\frac{2}{3ab^2}}$; (6) $x^2\sqrt{\frac{y}{8x}}$;

   (7) $\sqrt{\frac{x^2-y^2}{a}}$ ($x>y$); (8) $(x-y)\sqrt{\frac{b^2}{x^2-y^2}}$ ($x>y$)

   (9) $\sqrt{(a^2-b^2)(a^4-b^4)}$ ($a>b$); (10) $\sqrt{a^{2n+1}b^3}$.

5. What is meant by like surds? Which of the following expressions are like surds?

   $$\sqrt{44},\ \sqrt{\frac{1}{x}},\ -\sqrt{1\frac{5}{11}},\ \sqrt{x^3y^2},\ \sqrt{175},\ 2\sqrt{a^2x},\ \frac{1}{2}\sqrt{63},$$

   $$-\sqrt{99},\ 5\sqrt{3\frac{4}{7}},\ \sqrt{\frac{m}{1-2x+x^2}}\ (x>1),\ \sqrt{225m^3}.$$

6. Calculate:

   (1) $\left(\sqrt{24}-\sqrt{\frac{1}{2}}+2\sqrt{\frac{2}{3}}\right)-\left(\sqrt{\frac{1}{8}}+\sqrt{6}\right)$;

   (2) $7\sqrt{a}+5\sqrt{a^2x}-4\sqrt{\frac{b^2}{a}}-6\sqrt{\frac{b^2x}{9}}$;

   (3) $2\sqrt{12}\bullet\frac{1}{4}\sqrt{3}\div 5\sqrt{2}$;

   (4) $9\sqrt{45}\div 3\sqrt{\frac{1}{5}}\times\frac{3}{2}\sqrt{2\frac{2}{3}}$;

(5) $\left(6\sqrt{\frac{3}{2}}-5\sqrt{\frac{1}{2}}\right)\left(\frac{1}{4}\sqrt{8}-\sqrt{\frac{2}{3}}\right)$;

(6) $(\sqrt{2x}-3\sqrt{8x^3})\div 8\sqrt{\frac{x}{4}}$;

(7) $(10\sqrt{48}-6\sqrt{27}+4\sqrt{12})\div\sqrt{6}$;

(8) $(2\sqrt{3}+3\sqrt{6})(2\sqrt{3}-3\sqrt{6})$;

(9) $(\sqrt{x}+\sqrt{x-1})(\sqrt{x}-\sqrt{x-1})\quad (x>1)$;

(10) $(8\sqrt{5}+6\sqrt{3})^2$;

(11) $\left(\frac{3}{2}\sqrt{1\frac{2}{3}}-\sqrt{1\frac{1}{4}}\right)^2$;

(12) $(\sqrt{2}+2\sqrt{3}-3\sqrt{6})(\sqrt{2}-2\sqrt{3}+3\sqrt{6})$;

(13) $\frac{\sqrt{5}}{\sqrt{3}+1}+\frac{\sqrt{3}}{\sqrt{5}-\sqrt{3}}-\frac{2+\sqrt{3}}{2-\sqrt{3}}$;

(14) $\frac{4\sqrt{5}+3\sqrt{6}}{3\sqrt{5}-2\sqrt{6}}+\frac{\sqrt{5}}{\sqrt{5}+\sqrt{2}}$;

(15) $(5\sqrt{3}+2\sqrt{5})\div(2\sqrt{3}-\sqrt{5})$;

(16) $(\sqrt{2}+\sqrt{3}+\sqrt{5})(3\sqrt{2}+2\sqrt{3}-\sqrt{30})$;

(17) $\frac{n+2+\sqrt{n^2-4}}{n+2-\sqrt{n^2-4}}+\frac{n+2-\sqrt{n^2-4}}{n+2+\sqrt{n^2-4}}\quad (n>2)$.

7. When $x=2-\sqrt{3}$, find the value of the expression $(7+4\sqrt{3})x^2+(2+\sqrt{3})x+\sqrt{3}$.

8. Given $x=\frac{1}{2}(\sqrt{7}+\sqrt{5})$, $y=\frac{1}{2}(\sqrt{7}-\sqrt{5})$, find the value of the following expression:

(1) x^2-xy+y^2;　　(2) $\frac{x}{y}+\frac{y}{x}$.

9. Given $x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$, $x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$, where a, b, c are real numbers and $b^2-4ac\geq 0$, find the value of the following expression:

(1) x_1+x_2;　　(2) $x_1 \bullet x_2$;

(3) $ax_1^2+bx_1+c$;　　(4) $ax_2^2+bx_2+c$.

10. Solve the following equation:

(1) $\sqrt{6}(x+1)=\sqrt{7}(x-1)$;　　(2) $\frac{\sqrt{3}x}{\sqrt{2}}+1=\frac{2\sqrt{2}x}{\sqrt{3}}$.

11. Solve the following simultaneous linear equations:

(1) $\begin{cases}\sqrt{3}x-\sqrt{2}y=1\\ \sqrt{2}x-\sqrt{3}y=0\end{cases}$　　(2) $\begin{cases}\sqrt{2}x+\sqrt{3}y=\sqrt{7}\\ \sqrt{6}x-\sqrt{7}y=\sqrt{5}\end{cases}$

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