Chapter 11 Quadratic equations

I. Quadratic equation in one unknown

11.1 Quadratic equation in one unknown

We have learnt how to solve the linear equation in one unknown. Now let's have a look at the following question.

How can we cut a rectangular sheet of metal so that the area is 150 cm^2 and the length is 5 cm longer than the breadth?

In order to solve this problem, we have to find out the length and breadth of the rectangle. Let's suppose that the breadth is x cm. Then the length is $x + 5$ cm. According to the problem, we get

$$x(x+5)=150,$$

Remove the bracket, we get $x^2+5x=150$.

Both left hand side and the right hand side of the above equation is an **integral expression**. This kind of equation is called an integral equation. In this integral equation, there is only one unknown and the highest order of the unknown is 2. This kind of equation is called a quadratic equation in one unknown.

By rearranging the terms, the above equation can be written as

$$x^2+5x-150=0.$$

After rearranging, any quadratic equation in one unknown can be written in the form of $ax^2+bx+c=0$ ($a\neq 0$). This form is called the general form of quadratic equation in one unknown. The term ax^2 is called the second degree term where a is the coefficient of the second degree term. bx is called the first degree term where b is the coefficient of the first degree term. c is called the constant term. The coefficient of the first degree term, b and the constant term, c can be any real numbers. The coefficient of the second degree term, a is any real number not equal to 0. If a is equal to 0, then the equation is not a quadratic equation.

【Example】 Rewrite the equation $4x(x+3)=5(x-1)+8$ into general form. Write down the coefficient of the second degree, the coefficient of first degree and the constant tern.

Solution Removing the brackets, we get

$$4x^2+12x=5x-5+8,$$

Rearranging and grouping the terms, we get

$$4x^2+7x-3=0.$$

The coefficient of the second degree is 4, the coefficient of the first degree is 7 and the constant term is -3.

Practice

1. (*Mental*) Name the coefficient of the second degree, the coefficient of the first degree and the constant term of the equation
$$2x^2+x+4=0$$
2. Write down the coefficient of the second degree, the coefficient of the first degree and the constant term of the following equation:
(1) $4x^2+3x-2=0$; (2) $3x^2-5=0$.
3. Rewrite the following equations into general form and then write down the coefficient of the second degree, the coefficient of the first degree and the constant term.
(1) $3x^2=5x+2$; (2) $(x+3)(x-4)=-6$;
(3) $3x(x-1)=2(x+2)-4$; (4) $(2x-1)(3x+2)=x^2+2$;
(5) $(t+1)^2-2(t-1)^2=6t-5$;
(6) $(y+\sqrt{6})(y-\sqrt{6})+(2y+1)^2=4y-5$.

11.2 Solving quadratic equation in one unknown

1. Taking Square Root Method

Solve $x^2=4$.

Since x is the square root of 4 , so $x=\pm\sqrt{4}$.

Thus $x_1=2$ and $x_2=-2$. [3]

The method for solving quadratic equation in one unknown is called **Taking square Root Method**.

【**Example 1**】Solve the equation $x^2-25=0$.

Solution Rearranging terms, we get

$$x^2=25,$$

So

$$x=\pm\sqrt{25},$$

Thus $x_1=5$ and $x_2=-5$.

【**Example 2**】Solve the equation $(x+3)^2=2$.

Analysis: In the equation $x+3$ is the square root of 2, so we can use the taking square root method.

Solution $x+3=\pm\sqrt{2}$,

Thus

$$x+3=\sqrt{2} \quad \text{or} \quad x+3=-\sqrt{2},$$

$\therefore$ $x_1=-3+\sqrt{2}$ and $x_2=-3-\sqrt{2}$.

Hence, if one side of a quadratic equation in one unknown consists of the square of an expression of the unknown and the other side is a non-negative number, then we can use the Taking Square Root Method to solve the equation.

Practice

Solve the following equation by taking square root method:

(1) $x^2=256$; (2) $4y^2=9$; (3) $16x^2-49=0$;

(4) $t^2-45=0$; (5) $(2x-3)^2=5$; (6) $(x+1)^2-12=0$.

[3] In general x_1, x_2 are used to denote the roots of the quadratic equation in one unknown.

2. Completing Square Method

We have solved $(x+3)^2=2$ before. Since $x+3$ is the square root of 2, we use taking square method to solve it. If we expand the left hand side of the equation

$$(x+3)^2=2$$

we get

$$x^2+6x+7=0,$$

Hence in order to solve

$$x^2+6x+7=0,$$

we can try to change it into

$$(x+3)^2=2$$

Moving the constant term 7 of the equation $x^2+6x+7=0$ to the right hand side , then

$$x^2+6x=-7.$$

In order to make the left hand side into a perfect square, add the square of half of the coefficient of the first degree term to both sides.

$$x^2+6x+3^2=-7+3^2$$

$$(x+3)^2=2$$

Solving the equation, we get

$$x+3=\pm\sqrt{2},$$

So

$$x=-3\pm\sqrt{2},$$

thus

$$x_1=-3+\sqrt{2} \quad \text{and} \quad x_2=-3-\sqrt{2}.$$

This is called **Completing Square Method**. In this method, we first move the constant term to the right hand side. Then add terms to both hand sides so that the left hand side becomes a perfect square. If the right hand side is non-negative, we can solve the equation by taking square method.

【**Example 3**】Solve the equation $x^2-4x-3=0$.

Solution Changing terms, we get

$$x^2 - 4x = 3$$

Completing square, we get

$$x^2 - 4x + (-2)^2 = 3 + (-2)^2$$

$$(x-2)^2 = 7$$

Taking square roots, we get

$$x - 2 = \pm\sqrt{7},$$

$\therefore \quad x = 2 \pm \sqrt{7}$

Thus $x_1 = 2 + \sqrt{7}$ and $x_2 = 2 - \sqrt{7}$.

【**Example 4**】Solve the equation $2x^2 + 5x - 1 = 0$.

Analysis: The coefficient of the second degree is 2, we can divide both sides by 2 so that the coefficient of the second degree becomes 1. Then it will be easier to make the left hand side a perfect square.

Solution Divide both sides by 2, we get

$$x^2 + \frac{5}{2}x - \frac{1}{2} = 0$$

Move the constant term to the right hand side, we get

$$x^2 + \frac{5}{2}x = \frac{1}{2}$$

Completing square, we get

$$x^2 + \frac{5}{2}x + \left(\frac{5}{4}\right)^2 = \frac{1}{2} + \left(\frac{5}{4}\right)^2$$

$$\left(x + \frac{5}{4}\right)^2 = \frac{33}{16}$$

Taking square root on both sides, we get

$$x + \frac{5}{4} = \pm\frac{\sqrt{33}}{4},$$

$$\therefore \quad x = -\frac{5}{4} \pm \frac{\sqrt{33}}{4} = \frac{-5 \pm \sqrt{33}}{4}$$

Thus,

$$x_1 = \frac{-5+\sqrt{33}}{4} \text{ and } x_2 = \frac{-5-\sqrt{33}}{4}.$$

Practice

1. Fill in the space with appropriate number.

   (1) $x^2 + 6x + \quad = (x + \quad)^2$; (2) $x^2 - 4x + \quad = (x - \quad)^2$;

   (3) $x^2 + 3x + \quad = (x + \quad)^2$; (4) $x^2 - \frac{5}{2}x + \quad = (x - \quad)^2$;

   (5) $x^2 + px + \quad = (x + \quad)^2$; (6) $x^2 + \frac{b}{a}x + \quad = (x + \quad)^2$.

2. Solve the following equations by completing square method:

   (1) $x^2 - 6x + 4 = 0$; (2) $2t^2 - 7t - 4 = 0$;

   (3) $3x^2 - 1 = 6x$.

3. Quadratic Formula

Now let's use completing square method to solve the quadratic equation in one unknown.

$$ax^2 + bx + c = 0 \quad (a \neq 0).$$

Since $a \neq 0$, we can divide the whole equation by a and get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Move the constant term to the right hand side, so

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Add the square of half of the coefficient of first degree term to both sides, we get

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2,$$

Thus

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Since $a \neq 0$, so $4a^2 > 0$. When $b^2 - 4ac \geq 0$, we get

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}} = \pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

Thus

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

Hence

The Quadratic Formula to find the roots of the quadratic equation in one unknown $ax^2 + bx + c = 0$ ($a \neq 0$) is

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \quad (b^2 - 4ac \geq 0)$$

We can see that the roots of the quadratic equation in one unknown is determined by the coefficients *a*, *b* and *c*. So when we solve the quadratic equation in one unknown, we first transform the equation into the general form, then substitute the values of the coefficients *a*, *b* and *c* into the quadratic formula to find the roots of the equation and this method is called **the Quadratic Formula Method**.

【Example 5】 Solve the equation $2x^2 + 7x - 4 = 0$.

Solution $a = 2$, $b = 7$ and $c = -4$.

$$b^2 - 4ac = 7^2 - 4 \times 2 \times (-4) = 81,$$

$$x = \frac{-7 \pm \sqrt{81}}{2 \times 2} = \frac{-7 \pm 9}{4}$$

$$\therefore \quad x_1 = \frac{-7+9}{4} = \frac{1}{2}, \quad x_2 = \frac{-7-9}{4} = -4.$$

【Example 6】 Solve the equation $x^2 + 2 = 2\sqrt{2}x$.

Solution Rearranging the terms, we get

$$x^2 - 2\sqrt{2}x + 2 = 0$$

$a = 1$, $b = -2\sqrt{2}$ and $c = 2$.

$$b^2 - 4ac = (-2\sqrt{2})^2 - 4 \times 1 \times 2 = 0,$$

$$x = \frac{2\sqrt{2} \pm 0}{2} = \sqrt{2}$$

$$\therefore \quad x_1 = x_2 = \sqrt{2}.$$

Note: This equation has two equal real roots.

【Example 7】 Solve the equation $x^2 + x - 1 = 0$ (give the answer correct to 0.001).

Solution $a = 1$, $b = 1$ and $c = -1$.

$$b^2 - 4ac = 1^2 - 4 \times 1 \times (-1) = 5,$$

$$x = \frac{-1 \pm \sqrt{5}}{2}.$$

Using the calculator we get $\sqrt{5} = 2.236$, so

$$x_1 = \frac{-1+2.236}{2} = 0.618, \quad x_2 = \frac{-1-2.236}{2} = -1.618.$$

This value is called the G**olden Ratio**.

【Example 8】 Solve for *x* in the equation

$$x^2 - a(3x - 2a + b) - b^2 = 0.$$

Solution Rearranging the equation, we get

$$x^2 - 3ax + (2a^2 - ab - b^2) = 0$$

The coefficient of x^2 is 1. The coefficient of x is $-3a$ and the constant term is $2a^2 - ab - b^2$.

$$(-3a)^2 - 4 \times 1 \times (2a^2 - ab - b^2) = a^2 + 4ab + 4b^2$$
$$= (a + 2b)^2$$

$$x=\frac{3a\pm\sqrt{(a+2b)^2}}{2}=\frac{3a\pm(a+2b)}{2}$$

$$\therefore \quad x_1=\frac{3a+a+2b}{2}=2a+b\,,\quad x_2=\frac{3a-a-2b}{2}=a-b\,.$$

Practice

1. Rewrite the following equation into the form $ax^2+bx+c=0$ and write down the values of *a*, *b* and *c*.
   (1) $x^2+9x=6$; (2) $2x^2+1=7x$;
   (3) $5x^2=3x+2$; (4) $8x=3x^2-1$.
2. Use the formula to solve the following equation
   (1) $2x^2+5x-3=0$; (2) $6x^2-13x-5=0$;
   (3) $2y^2-4y-1=0$; (4) $t^2+2t=5$;
   (5) $p(p-8)=16$; (6) $\frac{5}{2}y^2+2y=1$;
   (7) $0.3x^2+x=0.8$; (8) $x^2+3=2\sqrt{3}x$.
3. Use the formula to solve the following equation and give the answer of the roots correct to 0.01.
   (1) $x^2+3x-5=0$; (2) $x^2-6x+4=0$.
4. Solve for the value of *x* in the following equation.
   (1) $2x^2-mx-m^2=0$;
   (2) $abx^2-(a^2+b^2)x+ab=0$ ($ab\neq 0$).

4. Factorization Method

As we know, we can solve the quadratic equation in one unknown by formula. For some equations with particular coefficients, e.g. $x^2=4$, we can simply solve by taking square root. Now, we try to learn another simple method called the Factorization Method.

For example. for the equation $x^2=4$, besides using the taking square method, we can solve it by the following method.

Rearranging terms, we get $x^2-4=0$.

We can factorize the left hand side into the product of two linear factors. Thus $x^2-4=(x-2)(x+2)$.

Hence the equation becomes

$$(x-2)(x+2)=0\,.$$

When the product of two factors is equal to 0, then at least one of the factor is equal to 0. Conversely, if one of the two factors is equal to 0, then their product is equal to 0. e.g. To make $(x-2)(x+2)$=0, it is necessary and sufficient to have $x-2=0$ or $x+2=0$. Hence solving the equation

$$(x-2)(x+2)=0$$

is the same as solving the equation $x-2=0$ or $x+2=0$ In solving these two linear equation, we get $x=2$ or $x=-2$.

Thus the roots of the original equation $x^2=4$ are

$$x_1=2 \text{ and } x_2=-2\,.$$

This method of solving quadratic equation in one unknown is called **Factorization Method**. When using this method, it is required to transform the equation so that one side of the equation is zero and that the other side can be factorized into linear factors.

【Example 9】 Solve the equations: (1) $x^2-3x-10=0$;
(2) $(x+3)(x-1)=5$.

Solution (1) Factorizing the left hand side, we get

$$(x-5)(x+2)=0$$
$$x-5=0 \text{ or } x+2=0$$
$$\therefore \quad x_1=5,\ x_2=-2\,.$$

(2) The original equation can be written as

$$x^2+2x-3=5,$$

Thus

$$x^2+2x-8=0\,.$$

Factorizing the left hand side, we get

$$(x-2)(x+4)=0$$
$$x-2=0 \text{ or } x+4=0$$
$$\therefore \quad x_1=2 \text{ and } x_2=-4\,.$$

【**Example 10**】Solve the equations: (1) $3x(x+2)=5(x+2)$;

(2) $(3x+1)^2-4=0$.

Solution (1) The equation

$$3x(x+2)-5(x+2)=0.$$

Factorizing the left hand side, we get

$$(x+2)(3x-5)=0$$

$$x+2=0 \quad \text{or} \quad 3x-5=0$$

$$\therefore \quad x_1=-2 \quad \text{and} \quad x_2=\frac{5}{3}.$$

(2) Factorizing the left hand side, we get

$$[(3x+1)+2][(3x+1)-2]=0,$$

thus

$$(3x+3)(3x-1)=0.$$

$$x+1=0 \quad \text{or} \quad 3x-1=0$$

$$\therefore \quad x_1=-1 \quad \text{and} \quad x_2=\frac{1}{3}.$$

Practice

1. (*Mental*) What are the roots?
   (1) $x(x-2)=0$; (2) $(y+2)(y-3)=0$;
   (3) $(3x+2)(2x-1)=0$; (4) $x^2=x$.
2. Solve the following equation by factorization method
   (1) $5x^2+4x=0$; (2) $\sqrt{2}y^2=3y$;
   (3) $x^2+7x+12=0$; (4) $x^2-10x+16=0$;
   (5) $x^2+3x-10=0$; (6) $x^2-6x-40=0$;
   (7) $t(t+3)=28$; (8) $(x+1)(x+3)=15$.
3. Solve the following equation by factorization method:
   (1) $(y-1)^2+2y(y-1)=0$; (2) $6(x+5)=x(x+5)$;
   (3) $(2y-1)^2-9=0$; (4) $(3x+2)^2=4(x-3)^2$.

We can also use cross multiplication method to solve some quadratic equation with coefficient of second degree not equal to 1.

【**Example 11**】 Solve the equations: (1) $3x^2-16x+5=0$;

(2) $2x(4x+13)=7$.

Solution (1)

$$(3x-1)(x-5)=0$$

$$3x-1=0 \quad \text{or} \quad x-5=0$$

$$\therefore \quad x_1=\frac{1}{3} \quad \text{and} \quad x_2=5.$$

(2) Rearranging the equation.

$$8x^2+26x-7=0$$

$$(4x-1)(2x+7)=0$$

$$4x-1=0 \quad \text{or} \quad 2x+7=0$$

$$\therefore \quad x_1=\frac{1}{4} \quad \text{and} \quad x_2=-\frac{7}{2}.$$

Practice

Solve the equation (No. 1～2):

1. (1) $3x^2-7x+2=0$; (2) $2x^2-11x-21=0$;
   (3) $14x^2+3x-5=0$; (4) $15x^2-14x-8=0$.
2. (1) $6(2x^2+1)=17x$; (2) $2x(4x-7)=15$.
3. Solve for x from the following equation.
   (1) $5m^2x^2-17mx+14=0$; (2) $10a^2x^2-7abx+b^2=0$.

11.3 Discriminant of Roots of Quadratic Equation in One Unknown.

We have known that using completing square method we can transform any quadratic equation of one unknown $ax^2+bx+c=0$ ($a\neq 0$) into

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}.$$

Since $a \neq 0$, so we have

(1) When $b^2-4ac>0$, the right hand side of the equation is positive. So the equation has two distinct real roots

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}, \quad x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

(2) When $b^2-4ac=0$, the right hand side is equal to 0. So the equation has two equal real roots

$$x_1=x_2=-\frac{b}{2a}.$$

(3) When $b^2-4ac<0$, the right hand side of the equation is negative and $\left(x+\frac{b}{2a}\right)^2$ cannot be negative. So the equation has no real roots.

Hence, we know that the value of b^2-4ac determines the conditions of the roots of the quadratic equation in one unknown. So we call b^2-4ac the discriminant of roots of the equation $ax^2+bx+c=0$.

Generally we use the symbol Δ[4] to represent it.

Summing up:

(i) when $\Delta>0$, the quadratic equation $ax^2+bx+c=0$ has two distinct real roots;

(ii) when $\Delta=0$, the equation has two equal real roots;

(iii) when $\Delta<0$, the equation has no real roots.

【Example 1】 Without solving the equation, determine the condition of the roots of the following equations.

(1) $2x^2+3x-4=0$; (2) $16y^2+9=24y$;

(3) $5(x^2+1)-7x=0$.

Solution (1) $\because$ $\Delta=3^2-4\times2\times(-4)=9+32>0$

$\therefore$ the equation has two distinct real roots.

4 「Δ」 is a Greek alphabet read as delta.

(2) Rearranging the terms, we get

$$16y^2-24y+9=0$$

$\because$ $\Delta=(-24)^2-4\times16\times9=576-576=0$

$\therefore$ the equation has two equal real roots.

(3) the original equation is

$$5x^2-7x+5=0$$

$\because$ $\Delta=(-7)^2-4\times5\times5=49-100<0$

$\therefore$ the equation has no real roots.

【Example 2】 For the equation $2x^2-(4k+1)x+2k^2-1=0$, find the value of k so that the equation has

(1) two distinct real roots,

(2) two equal real roots, and

(3) no real roots？

Solution $\Delta=[-4(k+1)]^2-4\times2(2k^2-1)=8k+9$

(1) When $8k+9>0$, i.e. $k>-\frac{9}{8}$, the equation has two distinct real roots.

(2) When $8k+9=0$, i.e. $k=-\frac{9}{8}$, the equation has two equal real roots.

(3) When $8k+9<0$, i.e. $k<-\frac{9}{8}$, the equation has no real roots.

【Example 3】 Prove that $(m^2+1)x^2-2mx+(m^2+4)=0$ has no real roots.

Proof $\Delta=(-2m)^2-4(m^2+1)(m^2+4)$

$=4m^2-4m^4-20m^2-16$

$=-4(m^4+4m^2+4)$

$=-4(m^2+2)^2$

For any real number m, $(m^2+2)^2$ must be positive. So $-4(m^2+2)^2$ must be negative , i.e.

$$\Delta<0,$$

Therefore the equation $(m^2+1)x^2-2mx+(m^2+4)=0$ has no real roots.

Practice

1. Without solving the equation , determine the condition of the real roots:

   (1) $3x^2+4x-2=0$; (2) $2y^2+5=6y$;

   (3) $4p(p-1)-3=0$; (4) $x^2+5=2\sqrt{5}x$;

   (5) $\sqrt{3}x^2-\sqrt{2}x+2=0$; (6) $3t^2-2\sqrt{6}t+2=0$.

2. Find the value of m so that the equation $x^2-2(m+1)x+(m^2-2)=0$

   (1) has two distinct real roots?

   (2) has two equal real roots?

   (3) has no real roots?

3. Prove that the equation $x^2+(2k+1)x-k^2+k=0$ has two distinct real roots.

Exercise 5

Solve the following equation by taking square method. (For questions 1～ 3):

1. (1) $49x^2-81=0$; (2) $\frac{1}{4}y^2=0.01$.
2. (1) $0.2x^2-\frac{3}{5}=0$; (2) $(x+3)(x-3)=9$.
3. (1) $(3x+1)^2=2$; (2) $(2t+3)^2-5=0$.

Solve the following equations by completing square method (For questions 4～5):

4. (1) $x^2+2x-99=0$; (2) $y^2+5y+2=0$.
5. (1) $3x^2-1=4x$; (2) $2x^2+\sqrt{2}x-30=0$.
6. Use completing square method to solve for x from the equation $x^2+px+q=0$.
7. Use the formula method to solve the following equation:

   (1) $x^2+2x-2=0$; (2) $6x^2+4x-7=0$;

   (3) $2y^2+8y-1=0$; (4) $x^2-2.4x-13=0$;

   (5) $2x^2-3x+\frac{1}{8}=0$; (6) $\frac{3}{2}t^2+4t=1$;

   (7) $3y^2+1=2\sqrt{3}y$; (8) $x^2+2(\sqrt{3}+1)+2\sqrt{3}=0$.

8. Use the formula to solve the following equations and give the answers of the roots correct to 0.01:

   (1) $x^2-3x-7=0$; (2) $x^2-3\sqrt{2}x+2=0$.

9. Solve the following equation by factorization method:

   (1) $8x^2-\frac{1}{2}x=3x^2+\frac{1}{3}x$; (2) $\frac{1}{3}(y+3)^2=\frac{1}{2}(y+3)$;

   (3) $x^2+7x+6=0$; (4) $x^2-5x-6=0$;

   (5) $y^2-17y+30=0$; (6) $y^2-7y-60=0$;

   (7) $9(2x+3)^2-4(2x-5)^2=0$;

   (8) $(2y+1)^2+3(2y+1)+2=0$.

10. Use any method to solve the following equations:

    (1) $x^2-3x+2=0$; (2) $x^2-3x-2=0$;

    (3) $x^2+12x+27=0$; (4) $(x-1)(x+2)=70$;

    (5) $(3-t)^2+t^2=9$; (6) $(y-2)^2=3$;

    (7) $(2x+3)^2=3(4x+3)$; (8) $(y+\sqrt{3})^2=4\sqrt{3}y$.

    (9) $(2x-1)(x+3)=4$; (10) $(y+1)(y-1)=2\sqrt{2}y$;

    (11) $x^2-\sqrt{3}x-\sqrt{2}x+6=0$ (12) $3x(x-1)=2-2x$.

11. Solve for x:

(1) $mx^2-(m-n)x-n=0 \quad (m\neq 0)$;

(2) $x^2-(2m+1)x+m^2+m=0$;

(3) $(x+a)(x-b)+(x-a)(x+b)=2a(ax-b)$;

(4) $abx^2-(a^4+b^4)x+a^3b^3=0 \quad (ab\neq 0)$.

12. Given $y=x^2-2x-3$. Find the value of x so that the value of y is 0. Find the value of x so that the value of y is -4.

13. Find the value of x so that the value of x^2+6x+5 is equal to the value of $x-1$.

14. Given $x^2-7xy+12y^2=0$. Show that $x=3y$ or $x=4y$.

15. Without solving the equations, determine the conditions of the roots of the following equations:

(1) $2x^2+4x+35=0$; (2) $4m(m-1)+1=0$;

(3) $0.2x^2-5=\frac{3}{2}x$; (4) $4(y^2+0.9)=2.4y$;

(5) $\frac{1}{2}x^2-\sqrt{2}=\sqrt{3}x$; (6) $2t=\sqrt{5}\left(t^2+\frac{1}{5}\right)$.

16. Find the value of m so that the equation

$$x^2+(2m+1)x+(m-2)^2=0$$

(1) has two distinct real roots?

(2) has two equal real roots?

(3) has no real roots?

17. Find the value of k so that the equation $4x^2-(k+2)x+k-1=0$ has two equal real roots and find the roots.

18. Show that the equation $(x-1)(x-2)=k^2$ has two distinct real roots.

11.4 Application of quadratic equation in one unknown

【Example 1】 The product of two consecutive odd numbers is 323. Find the two numbers.

Solution Let the smaller odd number be x. Then the other odd number is $x + 2$. According to the problem, we get

$$x(x+2)=323,$$

Rearranging, we get

$$x^2+2x-323=0.$$

Solving the equations we get

$$x_1=17,\quad x_2=-19.$$

Since odd numbers can be positive or negative, therefore both $x=17$, $x=-19$ satisfy the problem.

When $x=17$, $x+2=19$;

When $x=-19$, , $x+2=-17$.

Answer: The two odd numbers are 17 and 19, or -19 and -17.

Try a different approach: Let the smaller odd number be $x-1$. Then the other is $x+1$. How would the problem be solved?

【Example 2】 In the Diagram 11-1, the metal sheet is 80 cm in length and 60 cm in width. Four identical squares are cut from the four corners. Then the sides are folded to form a rectangular open box with base area 1500 cm^2. What is the length of side of the square cut?

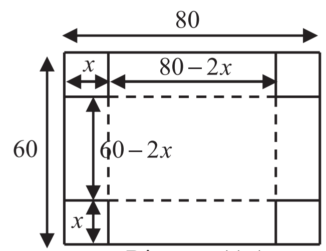


Diagram 11-1

Solution Let the length of the square be x cm. Then the length and the width of base of the box are $80-2x$ cm and $60-2x$ cm respectively. According to the problem, we get

$$(80-2x)(60-2x)=1500.$$

Rearranging, we get

$$x^2-70x+825=0.$$

Solving the equation, we get

$$x_1 = 15 \quad \text{and} \quad x_2 = 55.$$

When $x = 15$, $80 - 2x = 50$, $60 - 2x = 30$;
When $x = 55$, $80 - 2x = -30$, $60 - 2x = -50$
Since the length and the width of the base cannot be negative, so we take $x = 15$ only.

Answer: The length of the square cut is 15 cm.

【Example 3】 The production of a steel factory in last January is 5000T. The production last March increased to 7200T. What is the average monthly increase percentage in these two months ?

Analysis: Let the average monthly increase percentage be x. Then the production last February is $(5000 + 5000x)$ T, i.e. $5000(1+x)$ T. The production last March is $[5000(1+x) + 5000(1+x)x]$ T, i.e. $5000(1+x)^2$ T. According to the problem we can list the equation.

Solution Let the average monthly increase percentage be x. According to the problem, we get

$$5000(1+x)^2 = 7200,$$

Solving,

$$(1+x)^2 = 1.44$$

$\therefore$ $$1 + x = \pm 1.2$$

Thus

$$x_1 = 0.2, \quad x_2 = -2.2.$$

$x = -2.2$ does not satisfy the problem , so take $x = 0.2 = 20\%$ only.

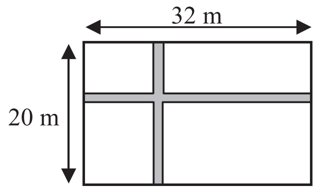
Answer: The average monthly increase percentage is 20%.

Practice

1. The product of two consecutive integers is 210. Find the two integers.
2. The sum of two numbers is 12 and the product is 23. Find the two numbers.

Practice

3. Solve the problem mentioned in the first section of this chapter.
4. The capacity of a rectangular box is 50 cm^3. The height is 6 cm and the length of the base is 5 cm longer than the width. What are the length and width of the base (Correct the answers to 0.1cm)?
5. Refer to the figure, the width of the rectangular land is 20 m and the length is 32 m. Two perpendicular roads of the same width are built. The remaining area is for farming and area is 540 m^2. What is the width of the roads ?



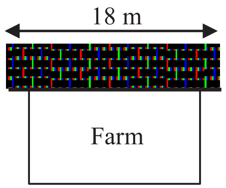
(No. 5)

6. In a farm, the production of the rice increases from 6 million kg to 7.26 million kg in two years. What is the average yearly increase percentage?
7. Mr. Cheung borrowed $500,000 from a bank. The interest is compounded yearly. After two years, he has to return $1620,000 to the bank. What is the rate of interest per annum?

Exercise 6

1. The difference of two number is 4 and the product is 16. Find the two numbers.
2. The value of a two digit number is equal to the square of the unit digit. The unit digit is bigger than the tens digit by 3. Find the number.
3. From a rectangular metal sheet of length 300 cm and width 200 cm, a smaller rectangular sheet is cut so that the length of the remaining rectangular frame is the same throughout. The area cut is half of the area of the original rectangular sheet. Find the width of the frame left (correct the answer to 1 cm).

4. A duct is to be built in a farm. The length of the duct is 750 m. the cross section is a trapezium of area $1.6m^2$. The upper length of the trapezium is 2 m longer than the depth of the duct. the lower length of the trapezium is 0.4m longer than the depth of the duct.
   (1) What is the upper length and lower length of the trapezium?
   (2) If $48m^3$ of the soil is dug each day, how many days are required to dig the duct?
5. Refer to the figure, the area of the rectangular farm is 150 m^2 .On one side of the farm is a wall of 18 m. The other three sides are fenced. The total length of the fence is 35 m. What is the length and width of the farm in m?



(No. 5)

6. The cost of production of an article was originally $300. After applying two cost reductions at equal rate for two periods, the new cost is $195. What is the rate of reduction for each time period (correct the answer to 1%)?
7. A factory plans to increase the production by 80% in two years. If the percentage increase per year must remain the same in these two years, what is the requreid percentage increase per year (correct the answer to 1%)?
8. In January a printing factory printed 500,000 copies of a book. In the first season, 1,750,000 copies were printed. What is the average monthly percentage increase in February and March (correct the answer to 1%)?

II. Relationship between Roots and Coefficients of Quadratic Equation in One Unknown

11.5 Relationship between Roots and Coefficients of Quadratic Equation in One Unknown

In solving quadratic equation in one unknown, we observe some relationship between the roots and the coefficients of the quadratic equation.

For example in solving $x^2-5x+6=0$, we get

$$x_1=2 \quad \text{and} \quad x_2=3.$$

It can be seen that $x_1+x_2=5$, which is equal to the negative of the coefficient of first degree, -5 while $x_1 \bullet x_2=6$ is equal to the constant term.

Taking another example, in solving $2x^2+5x-3=0$, we get

$$x_1=\frac{1}{2} \quad \text{and} \quad x_2=-3.$$

It can be seen that:

(i) $x_1+x_2=-\frac{5}{2}$, which is equal to the negative of the quotient of the coefficient of first degree, 5 divided by the coefficient of the second degree, 2.

(ii) $x_1 \bullet x_2=-\frac{3}{2}$, is equal to the quotient of the constant term -3 divided by the coefficient of second degree, 2.

In general, for the quadratic equation in one unknown $ax^2+bx+c=0$ ($a \neq 0$),

$$\because \quad x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}, \quad x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad x_1+x_2=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}=\frac{-2b}{2a}=-\frac{b}{a}$$

$$x_1 \cdot x_2 = \frac{-b+\sqrt{b^2-4ac}}{2a} \cdot \frac{-b-\sqrt{b^2-4ac}}{2a}$$
$$= \frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}$$
$$= \frac{4ac}{4a^2}$$
$$= \frac{c}{a}$$

Hence, the roots and the coefficients of quadratic equation have the following relationship:

> **If the roots of $ax^2+bx+c=0$ ($a \neq 0$) are x_1 and x_2,**
> **then $x_1+x_2=-\frac{b}{a}$, $x_1 \cdot x_2 = \frac{c}{a}$.**

If the equation $ax^2+bx+c=0$ ($a \neq 0$) is transformed into

$$x^2+\frac{b}{a}x+\frac{c}{a}=0,$$

we can write it as $x^2+px+q=0$, where $p=\frac{b}{a}$ and $q=\frac{c}{a}$.

Thus **if the equation $x^2+px+q=0$ has two roots x_1, x_2, then $x_1+x_2=-p$, $x_1 \cdot x_2 = q$.**

【Example 1】 Given that one of the roots of the equation $5x^2+kx-6=0$ is 2, find the other root and the value of *k*.

Solution Let the other root be x_1, then $x_1 \cdot 2 = -\frac{6}{5}$,

$\therefore \quad x_1 = -\frac{3}{5}$

and $\quad \left(-\frac{3}{5}\right)+2=-\frac{k}{5}$

$\therefore \quad k=-5\left[\left(-\frac{3}{5}\right)+2\right]=-7$

Answer: The other root is $-\frac{3}{5}$ and $k=-7$.

Try a different approach: Substitute $x=2$ into the original equation to find the value of *k* first. Then find the other root of the quadratic equation.

【Example 2】 For the equation $2x^2+3x-1=0$, using the relation of the roots and the coefficients, find

(1) the sum of squares of the roots;

(2) the sum of the reciprocals of the roots.

Solution Let the roots of the equation be x_1 and x_2.

Then $x_1+x_2=-\frac{3}{2}$ and $x_1 \cdot x_2 = -\frac{1}{2}$.

(1) $\because \quad (x_1+x_2)^2 = x_1^2+2x_1x_2+x_2^2$

$\therefore \quad x_1^2+x_2^2=(x_1+x_2)^2-2x_1x_2$

$$=\left(-\frac{3}{2}\right)^2-2\times\left(-\frac{1}{2}\right)=\frac{13}{4}$$

(2) $\frac{1}{x_1}+\frac{1}{x_2}=\frac{x_1+x_2}{x_1x_2}=\frac{-\frac{3}{2}}{-\frac{1}{2}}=3.$

Answer: The sum of squares of the roots is $\frac{13}{4}$ and the sum of the reciprocals of the roots is 3.

Practice

1. (*Mental*) In the following equation, what are the sum and the product of the roots?

(1) $x^2-3x+1=0$; (2) $3x^2-2x-2=0$;

(3) $2x^2-9x+5=0$; (4) $4x^2-7x+1=0$;

(5) $2x^2+3x=0$; (6) $3x^2-1=0$.

Practice

2. Given that one of the roots of the equation $3x^2-19x+m=0$ is 1, find the other root and the value of *m*.
3. Given that x_1, x_2 are the roots of the equation $2x^2+4x-3=0$ use the relationship of the roots and the coefficients of the equation to find the value of the following expression:

   (1) $(x_1+1)(x_2+1)$; (2) $\frac{x_2}{x_1}+\frac{x_1}{x_2}$.

If two numbers x_1 and x_2 are the roots of a quadratic equation in one unknown, we can construct or compose a quadratic equation with roots x_1 and x_2 and with coefficient of second degree 1. Assume that the quadratic equation is

$$x^2+px+q=0,$$

Then from the relationship of the roots and the coefficients, we get

$$x_1+x_2=-p,\quad x_1\bullet x_2=q,$$

thus

$$p=-(x_1+x_2),\quad q=x_1\bullet x_2,$$

The quadratic equation $x^2+px+q=0$ becomes

$$x^2-(x_1+x_2)x+x_1\bullet x_2=0,$$

Thus, **the quadratic equation in one unknown (with coefficient of second degree to be 1) with two roots x_1, x_2 is**

$$\mathbf{x^2-(x_1+x_2)x+x_1\bullet x_2=0}.$$

【Example 3】 Find the quadratic equation of one unknown with roots: $-3\frac{1}{3}$ and $2\frac{1}{2}$.

Solution The required equation is

$$x^2-\left(-3\frac{1}{3}+2\frac{1}{2}\right)x+\left(-3\frac{1}{3}\right)\times2\frac{1}{2}=0,$$

That is

$$x^2+\frac{5}{6}x-\frac{25}{3}=0,$$

or

$$6x^2+5x-50=0.$$

【Example 4】 Given the sum of two numbers is 8 and the product is 9, find the two numbers.

Solution Since the sum of the two numbers is 8 and the product is 9, then the two numbers are the roots of the equation

$$x^2-8x+9=0.$$

Solving the equation, we get

$$x_1=4+\sqrt{7}\quad\text{and}\quad x_2=4-\sqrt{7}.$$

Hence the two numbers are $4+\sqrt{7}$ and $4-\sqrt{7}$.

【Example 5】 Given the equation $x^2-2x-1=0$, using the relationship of the roots and the coefficients, find a quadratic equation in one unknown with the roots equal to the cube of roots of the original equation.

Solution Let the roots of the equation $x^2-2x-1=0$ be x_1 and x_2. Then the roots of the required equation are x_1^3 and x_2^3.

$\because\quad x_1+x_2=2,\quad x_1\bullet x_2=-1$

$$\therefore\quad x_1^3+x_2^3=(x_1+x_2)(x_1^2-x_1x_2+x_2^2)$$
$$=(x_1+x_2)[(x_1+x_2)^2-3x_1x_2]$$
$$=2[2^2-3\times(-1)]$$
$$=14$$
$$x_1^3\bullet x_2^3=(x_1\bullet x_2)^3=(-1)^3=-1$$

Hence the required equation is

$$y^2-14y-1=0.$$

Practice

1. (*Mental*) Determine whether the numbers in the brackets are the roots of the equation in front:
   (1) $x^2+5x+4=0$ (1, 4); (2) $x^2-6x-7=0$ $(-1, 7)$;
   (3) $2x^2-3x+1=0$ $(\frac{1}{2}, 1)$ (4) $3x^2+5x-2=0$ $(-\frac{1}{3}, 2)$;
   (5) $x^2-8x+11=0$ $(4-\sqrt{5},\ 4+\sqrt{5})$;
   (6) $x^2-4x+1=0$ $(-2-\sqrt{3},\ -2+\sqrt{3})$.
2. Find a quadratic equation with one unknown with the following roots.
   (1) $4,\ -7$; (2) $1+\sqrt{3},\ 1-\sqrt{3}$.
3. Given the sum of two numbers is -6 and the product is 2, find the two numbers.
4. Using the relationship of the roots and the coefficients, find a quadratic equation in one unknown so that the roots are twice the roots of the equation $x^2+3x-2=0$.

11.6 Factorization of a Quadratic trinomial

The form of the polynomial ax^2+bx+c is called a quadratic trinomial in x. Here the coefficients a, b and c are known and $a\neq 0$. We have learnt how to factorize a quadratic trinomial in the form $x^2+(a+b)x+ab$. Now let us learn how to factorize the general form ax^2+bx+c.

When we solve the equation $2x^2-6x+4=0$, we first factorize the left hand side,

$$2(x^2-3x+2)=0$$

$$2(x-1)(x-2)=0$$

Thus the roots of the equation are:

$$x_1=1 \text{ and } x_2=2.$$

We can do it in the reverse.

If we use the formula to find the roots, x_1 and x_2 of

$$ax^2+bx+c=0$$

then from the relationship of the roots and the coefficients, we get

$$x_1+x_2=-\frac{b}{a},\quad x_1\times x_2=\frac{c}{a},$$

That is

$$\frac{b}{a}=-(x_1+x_2),\quad \frac{c}{a}=x_1\times x_2$$

$$\therefore\quad ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

$$=a[x^2-(x_1+x_2)x+x_1x_2]$$

$$=a(x-x_1)(x-x_2)$$

Hence to factorize ax^2+bx+c, we can use the formula to find the roots x_1 and x_2, then we can write the quadratic trinomial as:

$$ax^2+bx+c=a(x-x_1)(x-x_2).$$

【Example 1】 Factorize $4x^2+8x-1$.

Solution The roots of the equation $4x^2+8x-1=0$ are

$$x=\frac{-8\pm\sqrt{8^2-4\times4\times(-1)}}{2\times4}=\frac{-8\pm4\sqrt{5}}{8}=\frac{-2\pm\sqrt{5}}{2},$$

That is

$$x_1=\frac{-2+\sqrt{5}}{2},\quad x_2=\frac{-2-\sqrt{5}}{2}$$

$$\therefore\quad 4x^2+8x-1=4\left(x-\frac{-2+\sqrt{5}}{2}\right)\left(x-\frac{-2-\sqrt{5}}{2}\right)$$

$$=(2x+2-\sqrt{5})(2x+2+\sqrt{5})$$

【Example 2】 Factorize $2x^2-8xy+5y^2$.

Analysis: Take $-8y$ as the coefficient of x and $5y^2$ as the constant term. Then $2x^2-8xy+5y^2$ can be taken as a quadratic trinomial in x.

Solution Take $2x^2-8xy+5y^2=0$ as a quadratic equation in x. Then the roots are

$$x=\frac{8y\pm\sqrt{(8y)^2-4\times2\times(5y^2)}}{2\times2}=\frac{8y\pm2\sqrt{6}y}{4}=\frac{4\pm\sqrt{6}}{2}y$$

$$\therefore\quad 2x^2-8xy+5y^2=2\left(x-\frac{4+\sqrt{6}}{2}y\right)\left(x-\frac{4-\sqrt{6}}{2}y\right)$$

Practice

1. Factorize:

   (1) $x^2+20x+96$; (2) $6x^2-13xy+7y^2$.

2. Within the domain of real numbers, factorize the following quadratic trinomial:

   (1) x^2-5x+3; (2) $3x^2+4xy-y^2$.

Exercise 7

1. (1) If -5 is a root of $5x^2+bx-10=0$, find the other root and the value of b;

   (2) If $2+\sqrt{3}$ is a root of $x^2-4x+c=0$, find the other root and the value of c.

2. If x_1 and x_2 are the roots of the equation $2x^2-6x+3=0$, use the relationship of the roots and coefficients to find the value of the following expression:

   (1) $x_1^2x_2+x_1x_2^2$; (2) $\left(x_1+\frac{1}{x_2}\right)\left(x_2+\frac{1}{x_1}\right)$;

   (3) $(x_1+x_2)^2$; (4) $\frac{1}{x_1^2}+\frac{1}{x_2^2}$.

3. If x_1 and x_2 are the roots of the equation $ax^2+bx+c=0$, show

   (1) $x_1^2+x_2^2=\frac{b^2-2ac}{a^2}$; (2) $\frac{1}{x_1}+\frac{1}{x_2}+\frac{b}{c}=0$.

4. Find a quadratic equation of one unknown with the following roots:

   (1) $-1,\ \frac{3}{4}$; (2) $\frac{-1+\sqrt{5}}{2},\ \frac{-1-\sqrt{5}}{2}$.

5. Find two numbers such that:

   (1) the sum is -5 and the product is -14;

   (2) the sum is $\sqrt{2}$ and the product is $-\frac{1}{4}$.

6. Using the relationship of the roots and the coefficients, find a quadratic equation in one unknown with roots equal to the following specifications

   (1) the negative of the roots of the equation $2x^2+3x-1=0$;

   (2) the reciprocal of the roots of the equation $2x^2+3x-1=0$;

   (3) the square of the roots of the equation $2x^2+3x-1=0$.

7. Factorize the following expression:

   (1) $5x^2+11x+6$; (2) $6y^2-13y+6$;

   (3) $-4x^2-4x+15$; (4) $10p^2-p-3$;

   (5) $a^2+40a+384$; (6) $3x^2y^2-10xy+7$.

8. Factorize the following expression within the domain of real numbers:

   (1) $2x^2-4x-5$; (2) $-3m^2-2m+4$;

   (3) $x^2-2\sqrt{2}x-3$; (4) $3x^2-5xy-y^2$.

III. Equation Convertible to Quadratic Equation in One Unknown

11.7 Simple high order equation

An integral equation containing one unknown with the highest order greater than 2 is called **a high degree equation in one unknown**. Some special form of high degree equation in one unknown can be transformed into a linear equation in one unknown or a quadratic equation in one unknown and be solved accordingly.

【Example 1】 Solve the equation $x^3 - 2x^2 - 15x = 0$.

Solution Factorize the left hand side of the equation,

$$x(x^2 - 2x - 15) = 0$$

$$x(x+3)(x-5) = 0$$

Thus $x = 0$, or $x + 3 = 0$, or $x - 5 = 0$.
So the roots of the equation are

$$x_1 = 0,\ x_2 = -3 \text{ and } x_3 = 5.$$

【Example 2】 Solve the equation $x^4 - 6x^2 + 5 = 0$.

Solution Let $x^2 = y$. Then $x^4 = y^2$ and the original equation becomes

$$y^2 - 6y + 5 = 0.$$

Solving this quadratic equation in y, we get

$$y_1 = 1 \text{ and } y_2 = 5.$$

When $y = 1$, $x^2 = 1$,
$\therefore$ $x = \pm 1$.
When $y = 5$, $x^2 = 5$,
$\therefore$ $x = \pm\sqrt{5}$.
Hence the four roots of the equation are

$$x_1 = 1,\ x_2 = -1,\ x_3 = \sqrt{5} \text{ and } x_4 = -\sqrt{5}.$$

Like the case in example 2, an equation of order 4 in one unknown which contains only even powers of the unknown is called a double quadratic equation. Usually this kind of equation can be solved by substituting the unknown x^2 by y, thereby transforming the equation into a quadratic equation in one unknown y.

【Example 3】 Solve the equation $(x^2 - x)^2 - 4(x^2 - x) - 12 = 0$.

Solution Let $x^2 - x = y$, the original equation becomes

$$y^2 - 4y - 12 = 0.$$

Solving the equation, we get

$$y_1 = 6,\ y_2 = -2.$$

When $y = 6$, $x^2 - x = 6$,

$$x^2 - x - 6 = 0.$$

Solving the equation, we get

$$x_1 = -2 \text{ and } x_2 = 3.$$

When $y = -2$, $x^2 - x = -2$,

$$x^2 - x + 2 = 0.$$

$\because$ $\Delta = (-1)^2 - 4\times 1\times 2 = -7 < 0$
$\therefore$ This equation has no real root.
Hence the original equation has two real roots:

$$x_1 = -2 \text{ and } x_2 = 3.$$

Practice

Solve the following equations:

1. (1) $x^3 - 8x^2 + 15x = 0$; (2) $x^3 + 7x^2 - 60x = 0$.
2. (1) $x^4 - 13x^2 + 36 = 0$; (2) $x^4 - 14x^2 + 45 = 0$;
   (3) $3x^4 - 2x^2 - 1 = 0$.
3. (1) $(x^2 + 2x)^2 - 14(x^2 + 2x) - 15 = 0$;
   (2) $(x^2 - 3x)^2 - 2(x^2 - 3x) - 8 = 0$.

11.8 Fractional Equation

We have learnt how to solve fractional equation in one unknown which can be transformed into a linear equation in one unknown. Now let us learn how to solve fractional equation in one unknown which can be transformed into quadratic equation in one unknown

The method is similar. We multiply both sides of the equation by an integral expression (formed by taking the L.C.M. of the denominators). The multiplication will simplify the denominators, and transform the fractional equation into an integral equation. The resultant integral equation may not be an equivalent equation to the original fractional equation, because some extraneous roots may be brought in the multiplication process. Hence, after solving the integral equation for the roots, we shall have to check whether these roots would satisfy the original fractional equation:

(i) We can check it by substituting the roots obtained one by one into the denominators of the fractional equation. If none of the denominator is equal to zero, then it is a root of the original equation. If some denominators become zero, then it is an extraneous root to be excluded.

(ii) It may be simpler to check it by substituting the roots obtained one by one into the integral expression used in the multiplication process. If the integral expression does not become zero, then the root is the root of the original equation. If the integral expression becomes zero, then it is an extraneous root to be excluded.

【Example 1】 Solve the equation $\frac{1}{x+2}+\frac{4x}{x^2-4}+\frac{2}{2-x}=1$.

Solution The original equation is

$$\frac{1}{x+2}+\frac{4x}{(x+2)(x-2)}-\frac{2}{x-2}=1,$$

Multiply both sides of the equation by $(x+2)(x-2)$ and simplify the denominators, we get

$$(x-2)+4x-2(x+2)=(x+2)(x-2),$$

Rearranging and simplifying, we get

$$x^2-3x+2=0,$$

Solving this equation, we get

$$x_1=1 \text{ and } x_2=2.$$

Checking:

Substitute $x=1$ into $(x+2)(x-2)$, it is not equal to zero

$\therefore$ $x=1$ is a root of the original equation

Substitute $x=2$ into $(x+2)(x-2)$, it is equal to zero,

$\therefore$ $x=2$ is an extraneous root to be excluded.

So the root of the original equation is 1.

【Example 2】 Solve the equation $\frac{2(x^2+1)}{x+1}+\frac{6(x+1)}{x^2+1}=7$.

Analysis: On the left hand side of the equation the fractions $\frac{x^2+1}{x+1}$ and $\frac{x+1}{x^2+1}$ are the reciprocal of each other. Hence we can use the method of substituting the unknown with a suitable variable.

Solution Let $\frac{x^2+1}{x+1}=y$. Then $\frac{x+1}{x^2+1}=\frac{1}{y}$. So the original equation becomes

$$2y+\frac{6}{y}=7.$$

Multiply both sides by y and simplify the denominators, we get

$$2y^2-7y+6=0.$$

Solving the equation, we get

$$y_1=2,\ y_2=\frac{3}{2}.$$

When $y=2$, $\frac{x^2+1}{x+1}=2$. Removing the denominator and simplifying , we get

$$x^2-2x-1=0.$$

$$\therefore \quad x=\frac{2\pm\sqrt{8}}{2}=1\pm\sqrt{2}.$$

When $y=\frac{3}{2}$, $\frac{x^2+1}{x+1}=\frac{3}{2}$. Removing the denominator and simplifying, we get

$$2x^2-3x-1=0.$$

$$\therefore \quad x=\frac{3\pm\sqrt{17}}{3}.$$

Checking:

Substitute respectively $x=1\pm\sqrt{2}$ and $x=\frac{3\pm\sqrt{17}}{3}$ into the denominators of the original equation and they are not equal to zero, so they are the roots of the original equation. Hence the roots of the original equation are

$$x_1=1+\sqrt{2},\quad x_2=1-\sqrt{2},$$

$$x_3=\frac{3+\sqrt{17}}{3} \text{ and } x_4=\frac{3-\sqrt{17}}{3}.$$

【Example 3】 Solve for x. $x+\frac{1}{x-1}=a+\frac{1}{a-1}$.

Solution Multiply both sides of the equation by $(a-1)(x-1)$, cancelling the denominator , we get

$$x(a-1)(x-1)+a-1=a(a-1)(x-1)+x-1.$$

Rearranging, we get

$$(a-1)x^2-a^2x+a^2=0.$$

Solving for x we get

$$x_1=a \text{ and } x_2=\frac{a}{a-1}.$$

Checking:

Substitute $x=a$ and $x=\frac{a}{a-1}$ respectively into the denominator $x-1$. It is not equal to zero. So they are both the roots of the original equation. Hence, the roots of the original equation are

$$x_1=a \text{ and } x_2=\frac{a}{a-1}.$$

Practice

1. Solve the equation:

   (1) $\frac{4}{x}-\frac{1}{x-1}=1$; (2) $\frac{2}{x}-\frac{3}{x+1}=2$;

   (3) $\frac{2}{1-x}=\frac{1}{1+x}+1$; (4) $\frac{2}{x^2-4}+\frac{x-4}{x^2+2x}=\frac{1}{x^2-2x}$.

2. Solve the equation by substituting the unknown with a suitable variable:

   (1) $\left(\frac{x}{x-1}\right)^2-5\left(\frac{x}{x-1}\right)+6=0$;

   (2) $\frac{3x}{x^2-1}+\frac{x^2-1}{3x}=\frac{5}{2}$.

3. Solve for x:

   (1) $x+\frac{1}{x}=c+\frac{1}{c}$;

   (2) $\frac{a-x}{b+x}=5-\frac{4(b+x)}{a-x}$ $(a+b\neq 0)$.

【Example 4】 Peter and Tom started walking 30 km from town A to the town B at the same time. Peter walked 2 km more than Tom in an hour. As a result, Peter arrived half an hour earlier than Tom. Find their speed.

Solution Let the speed of Tom be x km per hour. Then the speed of Peter is $(x+2)$ km per hour. According to the problem, we get

$$\frac{30}{x+2}=\frac{30}{x}-\frac{1}{2}.$$

Multiply both sides of the equation by $2x(x+2)$, cancelling the denominator and rearranging, we get

$$x^2+2x-120=0.$$

Solving the equation, we get

$$x_1=10,\ x_2=-12.$$

Checking confirms both $x_1=10$ and $x_2=-12$ are the roots of the original equation. But the speed cannot be a negative number, so we take the positive root $x=10$. Then $x+2=12$.

Answer: Peter's speed is 12 km per hour and Tom's speed is 10 km per hour.

【Example 5】 There are two pipes filling a tank. Pipe A takes 10 hours less than Pipe B to fill the tank. With both pipes working, it takes 12 hours to fill the tank. Working individually, how long does it take for Pipe A and Pipe B respectively to fill the tank ?

Analysis: Suppose Pipe B alone takes x hours to fill the tank. Then Pipe A takes $(x-10)$ hours to fill the tank . In an hour pipe B alone can fill $\frac{1}{x}$ of the tank while Pipe A can fill $\frac{1}{x-10}$ of the tank. We can list the equation as both pipes working would fill $\frac{1}{12}$ of the tank in an hour.

Solution Suppose Pipe B alone takes x hours to fill the tank. Then Pipe A takes $(x-10)$ hours to fill the tank . According to the problem, we get

$$\frac{1}{x-10}+\frac{1}{x}=\frac{1}{12}.$$

Multiply both sides of the equation by $12x(x-10)$, cancelling the denominator and rearranging, we get

$$x^2-34x+120=0.$$

Solving the equation, we get

$$x_1=30,\ x_2=4.$$

Checking confirms that both $x_1=30$ and $x_2=4$ are accepted as the roots of the original equation.

When $x=30$, $x-10=20$.

When $x=4$, $x-10=-6$.

Since the time for filling the tank cannot be negative, we only take the positive root $x=30$.

Answer: Pipe A alone takes 20 hours and Pipe B alone takes 30 hours to fill the tank.

Practice

1. A ditch of 960 m is to be built. If each day 20 m more is dug than as originally scheduled, then the task can be finished 4 days earlier. How many days are required in the original plan?
2. In a factory there is a stock of 350 T of coal. Due to an improvement in technology, there is a reduction of the consumption of 2 T of coal each day. Then the stock of coal can be used for 20 more days than scheduled. How many days is the stock planned to be used originally?
3. Two teams are going to carry out a green project. Team A and B, working together, take 6 days to finish the project. Team A alone takes 5 days less than Team B. How many days does team A and team B take respectively to finish the project?
4. Two teams A and B cooperated to finish a task. After working for 10 days, team A left for another job. Team B worked 2 more days to finish the task. If the task is done by one team only, Team A would take 4 days less than team B. How many days does each team alone take to finish the task?
5. A boat sailed downstream 24 km to a destination and then sailed upstream to return to the starting point. The whole trip took 3 hours and 20 minutes. The speed of the current was 3 km per hour. What was the speed of the boat in still water? What was the speed of the boat downstream and what was the speed of the boat upstream?

11.9 Irrational Equation

Let us look at the following equation:

$$8+x+\sqrt{64+x^2}=24.$$

In this equation, the unknown x is under a square root sign. Equation like this is called an **irrational equation** [5]. e.g. $x-\sqrt{x-1}=3$, $\sqrt{2x-4+1}=\sqrt{x+5}$, and $\sqrt{2x-3}+\dfrac{2}{\sqrt{2x-3}}=\sqrt{5x-1}$ etc. are irrational equations. But equations like $x^2+2\sqrt{2}x-1=0$ and $\dfrac{x}{\sqrt{2}-1}+\dfrac{1}{x-2}=1$ etc are not irrational equations. They are respectively integral equation and fractional equation.

Integral equation and fractional equation are all called **rational equations**.

Let us try to study the methods for solving irrational equations. For example, in solving the equation

$$\sqrt{2x^2+7x}-2=x.$$

We may move the term with the unknown inside the surd to one side of the equation and the rest to the other side. then we get

$$\sqrt{2x^2+7x}=x+2.$$

Squaring both sides, we get the rational equation

$$2x^2+7x=x^2+4x+4.$$

Rearranging, we get

$$x^2+3x-4=0.$$

Solving the equation, we get

$$x_1=1,\ \ x_2=-4.$$

Checking :

Substitute $x=1$ into the original equation,

Left hand side $=\sqrt{2\times1^2+7\times1}-2=\sqrt{2+7}-2=3-2=1$

Right hand side $=1$

$\therefore$ $x=1$ is a root of the original equation.

[5] Any equation containing letters under a square root sign is an irrational equation.

Substitute $x=-4$ into the original equation,

Left hand side $=\sqrt{2\times(-4)^2+7\times(-4)}-2=0$

Right hand side $=-4$

$\therefore$ $x=-4$ is an extraneous root.

Thus the root of the original equation is 1.

From the above example, we know that in solving irrational equations, when we transform the irrational equation into the rational equation, we square both sides to the same power. Then we may have introduced some extraneous roots. Thus after solving the rational equation for roots, we shall have to check whether the roots are solutions of the original irrational equation.

【Example 1】 Solve the equation $\sqrt{2x-4}-\sqrt{x+5}=1$.

Solution Rearrange terns, we get

$$\sqrt{2x-4}=1+\sqrt{x+5},$$

Squaring both sides, we get

$$2x-4=1+2\sqrt{x+5}+x+5.$$

i.e.

$$x-10=2\sqrt{x+5}.$$

Squaring both sides again, we get

$$x^2-20x+100=4(x+5).$$

So

$$x^2-24x+80=0.$$

Solving the equation, we get

$$x_1=4 \text{ and } x_2=20.$$

Checking :

Substitute $x=4$ into the original equation.

Left hand side $=\sqrt{2\times4-4}-\sqrt{4+5}=2-3=-1$

Right hand side $=1$

$\therefore$ $x=4$ is an extraneous root.

Substitute $x=20$ into the original equation.

Left hand side $=\sqrt{2\times20-4}-\sqrt{20+5}=6-5=1$

Right hand side $=1$

$\therefore \quad x=20$ is a root of the original equation.

Thus the root of the original equation is $x=20$.

Note: In solving this equation, we rearrange the terms first so that the left hand side consists of only one term with the unknown in the surd. This is a simpler method.

【Example 2】 Solve the equation $2x^2+3x-5\sqrt{2x^2+3x+9}+3=0$.

Analysis: This equation can be transformed into

$$2x^2+3x+9-5\sqrt{2x^2+3x+9}-6=0.$$

Here we know $2x^2+3x+9$ is the square of $\sqrt{2x^2+3x+9}$

Let $\sqrt{2x^2+3x+9}=y$, then the original equation is transformed into a quadratic equation of y.

Solution Let $\sqrt{2x^2+3x+9}=y$. Then $2x^2+3x+9=y^2$ and $2x^2+3x=y^2-9$. So the original equation is transformed into

$$y^2-9-5y+3=0,$$

That is

$$y^2-5y-6=0.$$

Solving this equation , we get

$$y_1=-1,\ \ y_2=6.$$

When $y=-1$, $\sqrt{2x^2+3x+9}=-1$, Since the square root $\sqrt{2x^2+3x+9}$ cannot be a negative number, so there is no real solution for $\sqrt{2x^2+3x+9}=-1$.

When $y=6$, $\sqrt{2x^2+3x+9}=6$. Squaring both sides, we get

$$2x^2+3x+9=36,$$

That is

$$2x^2+3x-27=0.$$

Solving this equation, we get

$$x_1=3 \text{ and } x_2=-\frac{9}{2}.$$

Checking :

Substitute $x=3$, $x=-\frac{9}{2}$ into the original equation respectively, both satisfy the equation. So they are both the roots of the original equation.

Thus the roots of the original equation are

$$x_1=3,\ \ x_2=-\frac{9}{2}.$$

Practice

1. (*Mental*) Does the following equation have roots and why?

   (1) $\sqrt{x^2+3x+2}=-4$; (2) $\sqrt{x+1+3}=2$.

2. Solve the following equation:

   (1) $\sqrt{2x+3}=x$; (2) $\sqrt{2x+3}=-x$.

   (3) $x+\sqrt{x-2}=2$; (4) $x-\sqrt{x-2}=2$;

   (5) $\sqrt{1-x}+\sqrt{12+x}=5$; (6) $\sqrt{3x-2}+\sqrt{x+3}=3$.

3. Solve the following equation by substituting the unknown with other variable:

   (1) $x^2+8x+\sqrt{x^2+8x}=12$ (2) $x^2-3x-\sqrt{x^2-3x+5}=1$.

Exercise 8

1. Solve the following equation:

   (1) $(x+1)(x-2)(x+3)=0$; (2) $(2x+1)(x^2-5x+6)=0$;

   (3) $2x^3+7x^2-4x=0$; (4) $(x^2+2)(x+3)=6$;

   (5) $x^3-2x^2-5x+10=0$; (6) $x^3-16=4x(x-1)$.

2. Solve the following equation:

   (1) $x^4-25x^2+84=0$; (2) $4x^4-5x^2+1=0$;

   (3) $2x^4-19x^2+9=0$.

3. By substituting the unknown with other variable, solve the following equation:

(1) $(x+1)^4-10(x+1)^2+9=0$;

(2) $(6x^2-7x)^2-2(6x^2-7x)=3$;

(3) $(3x^2-2x+1)(3x^2-2x-7)+12=0$.

4. Solve the following equation:

(1) $\frac{x-1}{x^2-2x}-\frac{1}{x}=\frac{x}{x-2}$; (2) $\frac{x+1}{x^2+x}-\frac{1}{3x}=\frac{x+5}{3x-3}$;

(3) $\frac{x}{x+3}+\frac{x}{x-3}=\frac{18}{x^2-9}$; (4) $\frac{1}{1-x}-2=\frac{3x-x^2}{1-x^2}$;

(5) $\frac{1}{2-x}-1=\frac{1}{x-2}-\frac{6-x}{3x^2-12}$;

(6) $\frac{4}{x-5}+\frac{x-3}{12-x}=\frac{x-45}{x^2-17x+60}$.

5. By substituting the unknown with other varieble, solve the following equation:

(1) $\left(\frac{x}{x+1}\right)^2+5\left(\frac{x}{x+1}\right)+6=0$;

(2) $\frac{8(x^2+2x)}{x^2-1}+\frac{3(x^2-1)}{x^2+2x}=11$; (3) $x^2+x+1=\frac{2}{x^2+x}$.

6. Solve the following equation:

(1) $\sqrt{x^2-5}=x-1$; (2) $2(\sqrt{x-3+3})=x$;

(3) $\sqrt{(x-3)(x-4)}-2\sqrt{3}=0$; (4) $\sqrt{x^2+4x-1}-\sqrt{x-3}=0$

(5) $\sqrt{2x+1}-\sqrt{x+2}=2\sqrt{3}$;

(6) $\sqrt{(x-1)(x-2)}+\sqrt{(x-3)(x-4)}=\sqrt{2}$.

7. By substituting the unknown with other variable, solve the following equation:

(1) $3x^2+15x+2\sqrt{x^2+5x+1}=2$;

(2) $x^2+3-\sqrt{2x^2-3x+2}=\frac{3}{2}(x+1)$;

(3) $\sqrt{\frac{x+2}{x-1}}+\sqrt{\frac{x-1}{x+2}}=\frac{5}{2}$.

8. Solve the equation mentioned in section 9 of this chapter.

9. Solve for x:

(1) $\frac{2x}{x-a}+\frac{12x^2}{a^2-x^2}=\frac{a-x}{x+a}$ ($a\neq 0$);

(2) $\sqrt{a-x}+\sqrt{x-b}=\sqrt{a-b}$.

10. Stations A and B are 150 m apart. An express train and a slow train depart from Station A to go to B at the same departure time. After 1 hour, the express train is 12 km ahead of the slow train. Finally the express train reached Station B 25 minutes earlier than the slow train. What are the speeds of the express train and the slow train respectively?

11. A ship travelled 46 km downstream and 34 km upstream. The total time taken is the same as the time required by the ship to travel 80 km in still water. Given that the speed of the current is 2 km per hour, find the speed of the ship in still water.

12. In a factory, 300 items are to be processed. After finishing 80 items, the technology is improved so that the factory can process 15 more items each day. As a result, the whole task is finished in 6 days. Find the number of items that can be processed each day after the improvement of the technology.

13. There are two pipes A and B that can fill a tank. Working individually, pipe A takes 15 hours less than pipe B to fill the tank. If pipe A works alone for 10 hours and then pipe B works alone for 15 hours, $\frac{2}{3}$ of the tank is filled. Working individually, how long does it take for pipe A and pipe B respectively to fill the tank?

IV. Simple Simultaneous Quadratic Equations in Two Unknowns

11.10 Simultaneous quadratic equations in two unknowns

Equation $x^2-2xy+y^2+x-y=6$ contains 2 unknowns and the highest order of the terms is 2. This kind of equation is called **quadratic equation in two unknowns**.

The general form of quadratic equation of x and y is

$$ax^2+bxy+cy^2+dx+cy+f=0.$$

ax^2, bxy, cy^2 are **the terms of order 2**. dx and cy are **the terms of order 1** and f is **the constant term**.

Let's look at the following system of equations:

$$\begin{cases} x^2+y^2=625 \\ y=x+5 \end{cases},$$

$$\begin{cases} x^2+2xy+y^2+x+y=12 \\ 2x^2+3xy+x+y=11 \end{cases}$$

The first simultaneous equations contains a quadratic equation in two unknowns and a linear equation in two unknowns. The second simultaneous equations contains two quadratic equations in two unknowns. Simultaneous equations like these are called **simultaneous quadratic equations in two unknowns**.

We shall study some methods to solve simple simultaneous quadratic equations in two unknowns.

11.11 Simultaneous equations with one quadratic equation in two unknowns and one linear equation in two unknowns.

Generally, this kind of simultaneous equations can be solved by substitution.

【Example 1】 Solve the simultaneous equations

$$\begin{cases} x^2-4y^2+x+3y-1=0 & (1) \\ 2x-y-1=0 & (2) \end{cases}$$

Solution From (2), we get

$$y=2x-1 \qquad (3)$$

Substitute (3) into (1), we get

$$x^2-4(2x-1)^2+x+3(2x-1)-1=0.$$

After rearranging, we get

$$15x^2-23x+8=0,$$

Solving the equation,

$$x_1=1 \text{ and } x_2=\frac{8}{15}.$$

Substituting $x_1=1$ into (3), we get

$$y_1=1.$$

Substituting $x_2=\frac{8}{15}$ into (3), we get

$$y_2=\frac{1}{15}.$$

Hence the roots of the original system of equations are:

$$\begin{cases} x_1=1 \\ y_1=1 \end{cases} \text{ and } \begin{cases} x_2=\frac{8}{15} \\ y_2=\frac{1}{15} \end{cases}.$$

【Example 2】 Solve the system of equation

$$\begin{cases} x+y=7 \\ xy=12 \end{cases}$$

Analysis: This simultaneous equations can be solved by substitutions. We can also solve it by treating the equations as the relationship between the roots and the coefficients of a quadratic equation. Accordingly, we can construct a quadratic equation in one unknown with the values x, y as its two roots, then solve this quadratic equation for the roots to obtain answers for x and y.

Solution In this system x and y are the roots of the quadratic equation

$$z^2 - 7z + 12 = 0.$$

Solving the equation, we get

$$z = 3 \text{ and } z = 4.$$

Hence the roots of the original system of equations are:

$$\begin{cases} x_1 = 3 \\ y_1 = 4 \end{cases} \text{and} \begin{cases} x_2 = 4 \\ y_2 = 3 \end{cases}.$$

Practice

1. Test whether the simultaneous equations

$$\begin{cases} x^2 + y^2 = 13 \\ x + y = 5 \end{cases}$$

has the following roots.

(1) $\begin{cases} x = 2 \\ y = 3 \end{cases}$ (2) $\begin{cases} x = 3 \\ y = 2 \end{cases}$

(3) $\begin{cases} x = 1 \\ y = 4 \end{cases}$ (4) $\begin{cases} x = -2 \\ y = -3 \end{cases}$

2. Solve the simultaneous equations:

(1) $\begin{cases} y = x + 5 \\ x^2 + y^2 = 625 \end{cases}$ (2) $\begin{cases} x^2 - 6x - 2y + 11 = 0 \\ 2x - y + 1 = 0 \end{cases}$

(3) $\begin{cases} x^2 + xy + y^2 + x + 5y = 0 \\ x + 2y = 0 \end{cases}$

3. Solve the simultaneous equations:

(1) $\begin{cases} x + y = 3 \\ xy = -10 \end{cases}$ (2) $\begin{cases} \frac{1}{x} + \frac{1}{y} = 5 \\ \frac{1}{xy} = 6 \end{cases}$

11.12 Simultaneous equations containing two quadratic equations in two unknowns.

For these simultaneous quadratic equations, we discuss only some special kinds of equations with simple solutions. Examples are given below:

【Example 1】 Solve the simultaneous quadratic equations

$$\begin{cases} x^2 + y^2 = 20 & (1) \\ x^2 - 5xy + 6y^2 = 0 & (2) \end{cases}$$

Analysis: Examine equation (2), the left hand side of equation (2) can be factorized $(x-2y)(x-3y)$ and the right hand side is zero. So equation (2) can be transformed into two linear equation $x - 2y = 0$ and $x - 3y = 0$. They can be grouped with equation (1) and form the systems

$$\begin{cases} x^2 + y^2 = 20 \\ x - 2y = 0 \end{cases}, \begin{cases} x^2 + y^2 = 20 \\ x - 3y = 0 \end{cases}.$$

Solving these two systems, the roots of the original system can be found.

Solution From (2), we get

$$(x-2y)(x-3y) = 0,$$

$\therefore\ x - 2y = 0$ 或 $x - 3y = 0$

So the original simultaneous quadratic equations can be transformed into two sets:

$$\begin{cases} x^2 + y^2 = 20 \\ x - 2y = 0 \end{cases}, \begin{cases} x^2 + y^2 = 20 \\ x - 3y = 0 \end{cases}.$$

Solving these two sets of simultaneous equations, the roots of the original simultaneous quadratic equations are

$$\begin{cases} x_1 = 4 \\ y_1 = 2 \end{cases}, \begin{cases} x_2 = -4 \\ y_2 = -2 \end{cases}, \begin{cases} x_3 = 3\sqrt{2} \\ y_3 = \sqrt{2} \end{cases}, \begin{cases} x_4 = -3\sqrt{2} \\ y_4 = -\sqrt{2} \end{cases}.$$

【**Example 2**】Solve the simultaneous equations

$$\begin{cases} x^2+2xy+y^2=9 & (1) \\ (x-y)^2-3(x-y)+2=0 & (2) \end{cases}$$

Analysis: Each of the two equations can be transformed into two linear equation in two unknowns. First group the first linear equation from equation (1) with the two linear equations from equation (2) to form two sets of equations. Then use the second linear equation from equation (1) and the two linear equations from equation (2) to form another two sets of equations. Thus we get four sets of equations, each containing two linear equations in two unknown. Solving these sets of equations for the unknowns, the solutions are the answers of the original simultaneous quadratic equations.

Solution From (1), we get

$$(x+y)^2=9.$$

$\therefore\ x+y=3$ 或 $x+y=-3$

From (2), we get

$$(x-y-1)(x-y-2)=0.$$

$\therefore\ x-y-1=0$ 或 $x-y-2=0$

Hence, the original system is transformed into the following sets of equations:

$$\begin{cases} x+y=3 \\ x-y-1=0 \end{cases},\ \begin{cases} x+y=3 \\ x-y-2=0 \end{cases},$$

$$\begin{cases} x+y=-3 \\ x-y-1=0 \end{cases},\ \begin{cases} x+y=-3 \\ x-y-2=0 \end{cases}.$$

Solving these four sets of equations, the roots of the original simultaneous quadratic equations are

$$\begin{cases} x_1=2 \\ y_1=1 \end{cases},\ \begin{cases} x_2=\frac{5}{2} \\ y_2=\frac{1}{2} \end{cases},\ \begin{cases} x_3=-1 \\ y_3=-2 \end{cases},\ \begin{cases} x_4=-\frac{1}{2} \\ y_4=-\frac{5}{2} \end{cases}.$$

Practice

1. Transform the following equation into two linear equations in two unknowns:

(1) $x^2-3xy+2y^2=0$; (2) $2x^2-5xy-3y^2=0$;

(3) $x^2-6xy+9y^2=16$; (4) $(x+y)^2-3(x+y)-10=0$;

(5) $x^2-4xy+4y^2-2x+4y=3$.

Solve the following simultaneous equations (For question 2 ~ 3):

2. (1) $\begin{cases} (x-y)(x-2y)=0 \\ 3x^2+2xy=20 \end{cases}$ (2) $\begin{cases} x^2+y^2=5 \\ 2x^2-3xy-2y^2=0 \end{cases}$

3. (1) $\begin{cases} (x-2y-1)(x-2y+1)=0 \\ (3x-2y+1)(2x+y-3)=0 \end{cases}$ (2) $\begin{cases} x^2+2xy+y^2=25 \\ 9x^2-12xy+4y^2=9 \end{cases}$

【**Example 3**】Solve the simultaneous equations

$$\begin{cases} x^2+3xy=28 & (1) \\ 2xy-y^2=7 & (2) \end{cases}$$

Analysis: In both equations, there are no first order terms, so we can eliminate the constant term and transform the simultaneous equations into an equation of the form $ax^2+bxy+cy^2=0$. Solving this equation with one of the original simultaneous quadratic equations, we obtain the answer to the original simultaneous quadratic equations.

Solution (1)－(2)×4, we get

$$x^2-5xy+4y^2=0$$

$$(x-y)(x-4y)=0$$

$\therefore\ x-y=0$ or $x-4y=0$

Thus, the original system can be transformed into two sets of equations.

$$\begin{cases} x-y=0 \\ 2xy-y^2=7 \end{cases},\ \begin{cases} x-4y=0 \\ 2xy-y^2=7 \end{cases}.$$

Solving these two sets of equations, we shall obtain roots of the original simultaneous quadratic equations:

$$\begin{cases}x_1=\sqrt{7}\\y_1=\sqrt{7}\end{cases},\ \begin{cases}x_2=-\sqrt{7}\\y_2=-\sqrt{7}\end{cases},\ \begin{cases}x_3=4\\y_3=1\end{cases},\ \begin{cases}x_4=-4\\y_4=-1\end{cases}.$$

【Example 4】 Solve the simultaneous quadratic equations

$$\begin{cases}x^2+y^2=5 & (1)\\xy=2 & (2)\end{cases}$$

Analysis: We can solve these simultaneous equations using the method similar to example (3). But for this problem, there is a simpler solution, By adding equation (1) to 2×equation (2), we obtain a new equation with the left hand side a perfect square form and the right hand side a constant. Taking square root of both sides, we get two linear equations. Similarly by computing equation (1) minus 2 x equation (2), we obtain two linear equations in two unknown. We can follow the working in example 2 to obtain four sets of linear equations in two unknowns. Solving these four sets of equations, solutions to the original simultaneous quadratic equations can be found.

Solution (1)+(2)×2, we get

$$(x+y)^2=9$$

$\therefore\ x+y=\pm3$ (3)

(1)−(2)×2, we get

$$(x-y)^2=1$$

$\therefore\ x-y=\pm1$ (4)

From (3) and (4), the original simultaneous quadratic equations can be transformed into four sets of simultaneous equations

$$\begin{cases}x+y=3\\x-y=1\end{cases},\ \begin{cases}x+y=3\\x-y=-1\end{cases},\ \begin{cases}x+y=-3\\x-y=1\end{cases},\ \begin{cases}x+y=-3\\x-y=-1\end{cases}.$$

Solving these four sets of simultaneous equations, the roots of the original simultaneous quadratic equations are

$$\begin{cases}x_1=2\\y_1=1\end{cases},\ \begin{cases}x_2=1\\y_2=2\end{cases},\ \begin{cases}x_3=-1\\y_3=-2\end{cases},\ \begin{cases}x_4=-2\\y_4=-1\end{cases}.$$

Practice

Solve the following simultaneous equations (Question 1 ~ 2):

1. (1) $\begin{cases}x^2-2xy-y^2=2\\xy+y^2=4\end{cases}$ (2) $\begin{cases}3x^2-y^2=8\\x^2+xy-y^2=4\end{cases}$

2. (1) $\begin{cases}x^2+y^2=20\\xy=8\end{cases}$ (2) $\begin{cases}x^2+y^2=13\\xy=-6\end{cases}$

3. (*Mental*) Given the simultaneous equations

$$\begin{cases}x^2+y^2=5 & (1)\\x^2-y^2=3 & (2)\end{cases}$$

Is the following solution correct? If it is wrong, how should it be corrected?

Solution [(1)+(2)]÷2, we get

$$x^2=4$$

$\therefore\ x=\pm2$

[(1)−(2)]÷2, we get

$$y^2=1$$

$\therefore\ y=\pm1$

The roots of the original simultaneous quadratic equations are

$$\begin{cases}x_1=2\\y_1=1\end{cases},\ \begin{cases}x_2=-2\\y_2=-1\end{cases}.$$

【Example 5】 Solve the simultaneous equations

$$\begin{cases}x^2-2y^2-y=1 & (1)\\2x^2-4y^2+x=6 & (2)\end{cases}$$

Analysis: In this system the coefficients of the order two terms in the two equations are in the same ratio (i.e. $\frac{1}{2}=\frac{-2}{-4}$). Taking (2) minus (1)×2, a linear equation is obtained. Solving the simultaneous equations containing this equation and any one of the equations in the original simultaneous equations, the roots of the original simultaneous equations can be obtained.

Solution (2)－(1)×4, we get

$$x+2y=4$$

Rearranging terms, we get

$$x=4-2y \qquad (3)$$

Substitute (3) into (1) and rearranging, we get

$$2y^2-17y+15=0$$

Solving this system, we get

$$y_1=1,\ y_2=\frac{15}{2}.$$

Substitute $y_1=1$ into (3), we get

$$x_1=2.$$

Substitute $y_2=\frac{15}{2}$ into (3), we get

$$x_2=-11.$$

So the roots of the original simultaneous equations are:

$$\begin{cases}x_1=2\\y_1=1\end{cases} \text{ and } \begin{cases}x_2=-11\\y_2=\frac{15}{2}\end{cases}.$$

【**Example 6**】Solve the simultaneous equations

$$\begin{cases}3x^2-9xy+2y^2-6x-3y+1=0 & (1)\\ x^2-3xy+y^2-2x-3y+3=0 & (2)\end{cases}$$

Analysis: In the two simultaneous equations, the coefficients of term x in the two equations are in the same ratio (i.e.

$\frac{3}{1}=\frac{-9}{-3}=\frac{-6}{-2}$). Use elimination method, an equation of y is obtained. Solving for y, and then substitute its value into any equation of the original simultaneous equations, x can be found.

Solution (2)×3－(1), we get

$$y^2-6y+8=0$$

Solving this equation, we get

$$y_1=2,\ y_2=4.$$

The original simultaneous equations can be transformed into two sets of equations:

$$\begin{cases}y=2\\x^2-3xy+y^2-2x-3y+3=0\end{cases}$$

$$\begin{cases}y=4\\x^2-3xy+y^2-2x-3y+3=0\end{cases}$$

Solving these two sets of equations, the roots of the original simultaneous equations are

$$\begin{cases}x_1=4+\sqrt{15}\\y_1=2\end{cases},\ \begin{cases}x_2=4-\sqrt{15}\\y_2=2\end{cases},$$

$$\begin{cases}x_3=7+\sqrt{42}\\y_3=4\end{cases},\ \begin{cases}x_4=7-\sqrt{42}\\y_4=4\end{cases}.$$

Practice

Solve the following simultaneous equations (Question 1～ 2):

1. (1) $\begin{cases}x+y+xy=5\\2x+y-xy=2\end{cases}$ (2) $\begin{cases}2x^2-xy-3x=0\\xy-2x^2-2y+1=0\end{cases}$

2. (1) $\begin{cases}2x^2+y^2+x-y=12\\x^2+y^2-y=6\end{cases}$ (2) $\begin{cases}2x^2-y^2+4x-2y=13\\x^2-y^2+2x+y=8\end{cases}$

Exercise 9

Solve the following simultaneous equations (Question 1～6):

1. (1) $\begin{cases} x+y+1=0 \\ x^2+4y^2=8 \end{cases}$ (2) $\begin{cases} (x-3)^2+y^2=9 \\ x+2y=0 \end{cases}$

(3) $\begin{cases} \dfrac{(x+1)^2}{9}-\dfrac{(y-1)^2}{4}=1 \\ x-y=1 \end{cases}$ (4) $\begin{cases} \dfrac{x^2}{5}+\dfrac{y^2}{4}=1 \\ y=x-3 \end{cases}$

2. (1) $\begin{cases} x+y=6 \\ xy=7 \end{cases}$ (2) $\begin{cases} \sqrt{x}+\sqrt{y}=3 \\ \sqrt{xy}=2 \end{cases}$

3. (1) $\begin{cases} \dfrac{4}{x-1}=\dfrac{5}{y+1}+1 \\ \dfrac{3}{x+3}=\dfrac{2}{y} \end{cases}$ (2) $\begin{cases} (x+2)(y-2)=xy \\ \sqrt{(x+1)(y+4)}+x+3=0 \end{cases}$

4. (1) $\begin{cases} x^2-2xy-3y^2=0 \\ y=\dfrac{1}{4}x^2 \end{cases}$ (2) $\begin{cases} x^2-4xy+3y^2=0 \\ x^2+y^2=10 \end{cases}$

(3) $\begin{cases} (x+y)^2-4(x+y)=5 \\ (x-y)^2-2(x-y)=3 \end{cases}$ (4) $\begin{cases} x^2+2xy+y^2=9 \\ (x-y)^2-3(x-y)-10=0 \end{cases}$

5. (1) $\begin{cases} x^2-xy+y^2=3 \\ 2x^2-xy-y^2=5 \end{cases}$ (2) $\begin{cases} x^2+y^2+xy=19 \\ x^2+y^2-xy=7 \end{cases}$

(3) $\begin{cases} x^2+y^2=10 \\ xy=5 \end{cases}$ (4) $\begin{cases} 4x^2+9y^2=25 \\ xy=2 \end{cases}$

6. (1) $\begin{cases} xy-x+y=7 \\ xy+x-y=13 \end{cases}$ (2) $\begin{cases} 2x^2-2xy+x=10 \\ x^2-xy+y=7 \end{cases}$

(3) $\begin{cases} x^2+y^2+x+y=18 \\ x^2-y^2+x-y=6 \end{cases}$ (4) $\begin{cases} x^2-3xy+2y^2+4x+3y=1 \\ 2x^2-6xy+y^2+8x+2y=3 \end{cases}$

7. (Note)[6] The area of a rectangular field is 864 sq. units. The sum of its length and its width is 60 units. What are the length and the width of the rectangular field?

8. If the width of a rectangle is increased by 1 cm, then the area is increased by 3 cm^2. Given the area of the original rectangle is 12 cm^2, find the length and the width of the rectangle.

9. Two teams working together can finish a project in 12 days. If team A works 5 days first, team B joins in, then the two teams take 9 more days to finish the project. How many days does each take to finish the project alone?

10. Two places A and B are 36 km apart. Some parts of the road are uphill and the rest are downhill. The speed of cycling downhill is 6 km per hour faster than the speed of cycling uphill. It takes 2 hours 40 minutes to cycle from place A to place B. It takes 20 minutes less to return from place B to place A. Find the speed of cycling uphill, downhill, the length of the uphill part and length of the downhill part from place A to place B.

[6] This problem is extracted from a book called《Field Plot Ratio Multiplication and Division Rule》written by a Sung Dynasty Mathematician named Yang in year 1275. The problem is: Area of field is 864 square units. Sum of width and length is 60 units. Find the width and the length of the field. Answer: width 24 units, length 36 units.

Chapter Summary

I. This chapter focuses on quadratic equation in one unknown and teaches the following aspects:

(i) methods for solving
(ii) application,
(iii) determinant,
(iv) relationship between roots and coefficients,
(v) convertible fractional equation,
(vi) convertible irrational equation,
(vii) simultaneous quadratic equations in two unknowns.

II. We learn how to solve integral equations (linear equation in one unknown, quadratic equation in two unknowns, high order equation), convertible fractional equation, convertible irrational equation and simultaneous linear equations in two unknowns and simultaneous quadratic equations in two unknowns. In solving these equations or simultaneous equations, we apply manipulations of addition, subtraction, multiplication, division, squaring and taking roots to simplify the problem leading to the solution. The rationale or line of thought in directing the manipulations leading to problem simplification is as follows:

(i) high order equation $\xrightarrow{\text{decreasing the power}}$ linear or quadratic equation;
(ii) fractional equation $\xrightarrow{\text{remove the denominator}}$ integral equation;
(iii) irrationl equation $\xrightarrow{\text{remove the square root}}$ rational equation;
(iv) equation of multiple unknowns $\xrightarrow{\text{eliminate unknowns}}$ equation of one uknown.

III. This chapter teaches four methods to solve quadratic equation in one unknown:

(i) taking square root method,
(ii) completing square method,

(iii) quadratic formula method
(iv) factorization method.

Among the four methods, the quadratic formula method is the most powerful as it can be applied to solve any quadratic equation in one unknown. However it may be advisable to analyze the circumstances of the problem because it may be simpler to apply one of the other methods to solve it.

IV. In a quadratic equation $ax^2 + bx + c = 0$, we can examine the discriminant to determine some useful information about the roots without having to solve the quadratic equation,

(i) when $b^2 - 4ac > 0$, there are two distinct real roots,
(ii) when $b^2 - 4ac = 0$, there are two identical real roots,
(iii) when $b^2 - 4ac < 0$, there is no real root.

V. In a quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$), if the roots are x_1 and x_2, we can determine the sum and product of the roots from the coefficients:

(i) $x_1 + x_2 = -\dfrac{b}{a}$ and

(ii) $x_1 \times x_2 = \dfrac{c}{a}$.

Thus, when the roots x_1 and x_2 are known, we can construct or compose the quadratic equation as

$$x^2 - (x_1 + x_2)x + x_1x_2 = 0.$$

VI. In solving fractional equation, both sides of the equation are multiplied by the L.C.M. of the denominators to transform the equation into an integral equation. In solving irrational equation, both sides are multiplied to equal power of some number to transform it into a rational equation. In multiplying these factors, extraneous roots may have been introduced into the equation. So checking of the result is required to ensure that the roots obtained are the roots of the original equation.

VII. In solving simultaneous quadratic equations in two unknown, we can apply the same methods as we do in solving simultaneous linear equations in two unknowns (such as substitution method, addition/subtraction method) For some special types of simultaneous equations, we can simplify the problem using substitution of variables or using factorization and hence solve the simpler equations.

Revision Exercise 11

1. Solve the following equation:
   (1) $(2x+1)^2+(x-2)^2-(2x+1)(x-2)=43$;
   (2) $x^2-(1+2\sqrt{3})x-3+\sqrt{3}=0$;
   (3) $(x^2-10)^2+3x^2=28$;
   (4) $(2x-3)^4-6(2x-3)^2+9=0$;
   (5) $(x^2-x)^2-4(2x^2-2x-3)=0$;
   (6) $(x^2+3x+4)(x^2+3x+5)=6$;
   (7) $(x+1)(x+2)(x+3)(x+4)+1=0$;
   (8) $\dfrac{3}{x^2-3x+2}-\dfrac{1}{x^2-x}=\dfrac{1}{x-2}+\dfrac{4}{x^2-2x}$;
   (9) $\dfrac{x+1}{(x+3)(x-1)}-\dfrac{2x-2}{(x+3)(2-x)}=\dfrac{6x}{(1-x)(x-2)}$;
   (10) $\dfrac{3}{x-2}-\dfrac{4}{x-1}=\dfrac{1}{x-4}-\dfrac{2}{x-3}$;
   (11) $x^2+3x-\dfrac{20}{x^2+3x}=8$;
   (12) $\left(\dfrac{x^2-1}{x}\right)^2-\dfrac{7}{2}\left(\dfrac{x^2-1}{x}\right)+3=0$;
   (13) $2\left(x^2+\dfrac{1}{x^2}\right)-3\left(x+\dfrac{1}{x}\right)=1$;

   (14) $\sqrt{x+5}+\dfrac{3}{\sqrt{x+5}}=\sqrt{3x+4}$;
   (15) $\sqrt{2x-5}+\sqrt{x-3}=\sqrt{3x+4}$;
   (16) $\sqrt{3x+1}+\sqrt{4x-3}-\sqrt{5x+4}=0$;
   (17) $2x^2-14x-3\sqrt{x^2-7x+10}+18=0$;
   (18) $\dfrac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}=2-x$.
2. In the following question, all letters represent positive numbers.
   (1) Given $S=\dfrac{\pi D^2}{2}+\pi Dh$, express D in terms of S, π and h.
   (2) In $Q=mq+\dfrac{mv^2}{R}$ ($Q>mq$), Q, m, q, R are known, find v.
   (3) In $mf=\left(\dfrac{m}{K}-1\right)\dfrac{1}{K}$, m and f are known, find K.
   (4) In $\dfrac{1}{R}=\dfrac{1}{r}-\dfrac{1}{r-r_1}$ ($r_1>4R$), R and r_1 are known, find r.
   (5) In $c=\sqrt{a^2+b^2}$, a and c are known, find b.
3. The roots of the equation $ax^2+bx+c=0$ are in the ratio of 2 ∶ 3. Prove that $6b^2=25ac$.
4. Use the relationship between the roots and the coefficients , find a quadratic equation so that the roots are the
   (1) negatives of the roots of $x^2+px+q=0$.
   (2) reciprocals of the roots of $x^2+px+q=0$.
   (3) k times of the roots of $x^2+px+q=0$.
   (4) squares of the roots of $x^2+px+q=0$.
5. Factorize the following expression:
   (1) $a^2-2a-120$; (2) $-6p^2+11p+10$;
   (3) $3x^2-xy-10y^2$; (4) $15a^2-8ac-12b^2$.

6. Within the domain of real numbers, factorize the following:

(1) $\sqrt{3}a^2-\sqrt{6}a-\sqrt{2}a+2$;

(2) $9m^4-\frac{1}{4}n^4$;

(3) x^4-x^2-6;

(4) $6x^4-7x^2-3$;

(5) $9x^2-12xy+y^2$;

(6) $5x^2y^2+xy-7$.

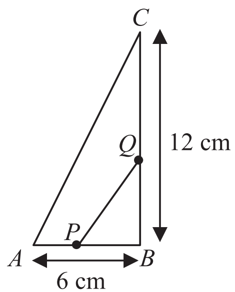
7. A metal wire of 100 cm is used to form a rectangular frame. Find the length of the frame in cm when the area of the frame is
(1) 500 cm^2; (2) 625 cm^2; (3) 800 cm^2.

8. A polygon has 35 diagonals. How many sides are there in the polygon?

9. (1) There are four consecutive numbers. Their sum is equal to the product of the largest number and the smallest number. Find the four numbers.

(2) There are three consecutive odd numbers. the sum of their squares is 251. Find the three numbers.

10. Refer to the figure, in $\triangle ABC$, $\angle B$=90°. Point P moves from A to B, along AB at a speed of 1 cm/s. Point Q moves from C to B, along BC at a speed of 2 cm/s. If P and Q starts from A and B respectively at same time, how many seconds later will the area of $\triangle PBQ$ be 8 cm^2?



(No. 10)

11. A container is filled with 63 L pure medicine. Some medicine is poured out and then the container is filled up with water. In the second time, same amount of diluted medicine is poured out and the container is filled up with water. Now the original pure medicine left in the container is 28L. How much liquid is poured out each time (in L)?

12. A container is filled up with an alkaline solution. 10 L of the solution is poured out and then the container is filled up again with water. In the second time, 10 L of the solution is poured out and then the container is again filled up with water. At that time the concentration of the solution is $\frac{1}{4}$ of that of the original solution. Find the capacity of the container.

13. Solve the following simultaneous equations:

(1) $\begin{cases} x^2-y^2-3x+2y=10 \\ x+y=7 \end{cases}$ (2) $\begin{cases} x+y=17 \\ x^2+y^2=169 \end{cases}$

(3) $\begin{cases} (x-2)^2+(y+3)^2=9 \\ 3x-2y=6 \end{cases}$ (4) $\begin{cases} 4x^2-9y^2=15 \\ 2x-3y=5 \end{cases}$

(5) $\begin{cases} x^2+y^2=5 \\ y^2=4x \end{cases}$ (6) $\begin{cases} x^2+y^2=101 \\ xy=-10 \end{cases}$

(7) $\begin{cases} (x-2)^2+(y-1)^2=25 \\ 2(x-2)^2-3(y-1)^2=5 \end{cases}$ (8) $\begin{cases} (x+3)^2+y^2=9 \\ 9(x-2)^2+4y^2=36 \end{cases}$

(9) $\begin{cases} x^2+y^2-3x-3y=8 \\ xy=10 \end{cases}$ (10) $\begin{cases} 2x^2-xy-y^2+3x+2y=3 \\ x^2-3x+2=0 \end{cases}$

(11) $\begin{cases} \frac{y}{x}+\frac{2x}{y}=3 \\ 2x+3y=4 \end{cases}$ (12) $\begin{cases} \frac{2}{x}+\frac{5}{y}=1 \\ \frac{4}{x^2}+\frac{25}{y^2}=25 \end{cases}$

(13) $\begin{cases} \sqrt{x+1}+\sqrt{y-2}=5 \\ x-y=12 \end{cases}$ (14) $\begin{cases} \sqrt{x+1}+\sqrt{y-1}=5 \\ x+y=13 \end{cases}$

(15) $\begin{cases} \sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{5}{2} \\ x+y=10 \end{cases}$ (16) $\begin{cases} xy=3 \\ yz=6 \\ xz=2 \end{cases}$

14. (1) Given $\begin{cases} x=1 \\ y=3 \end{cases}$, $\begin{cases} x=2 \\ y=-4 \end{cases}$ are the roots of the equation $(y+k)^2 = 2(x+h)$, find the values of h and k.;

(2) Given $\begin{cases} x=0 \\ y=0 \end{cases}$, $\begin{cases} x=2 \\ y=-4 \end{cases}$ are the roots of the equation $(x+h)^2+(y+k)^2=10$, find the values of h and k.

15. (1) Given $\begin{cases} x=1 \\ y=\frac{3}{2}\sqrt{3} \end{cases}$, $\begin{cases} x=\frac{2}{3}\sqrt{5} \\ y=-2 \end{cases}$ are the roots of the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, find the positive values of a and b;

(2) Given $\begin{cases} x=-5\sqrt{2} \\ y=2 \end{cases}$, $\begin{cases} x=6 \\ y=-\frac{2}{5}\sqrt{11} \end{cases}$ are the roots of the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, find the positive values of a and b.

16. (1) From the simultaneous equations

$$\begin{cases} x^2+y^2=8 \\ x+y=b \end{cases}$$

eliminate y and find the equation containing x only.;

(2) When $b=3$, how many real roots are there in the equation of x ? Find the number of real roots when $b=4$ and when $b=5$ respectively?

17. (1) Find the value of m for the simultaneous equations

$$\begin{cases} y^2=4x \\ y=2x+m \end{cases}$$

to have two identical real roots and find the roots;

(2) Find the value of m for the simultaneous equations

$$\begin{cases} x^2+2y^2=6 \\ mx+y=3 \end{cases}$$

to have two identical real roots and solve the simultaneous equations.

18. When will the simultaneous equations

$$\begin{cases} x+y=a \\ xy=b \end{cases}$$

(1) have two distinct real roots?
(2) have two identical real roots?
(3) have no real roots?

19. A and B work together and take 4 days to finish a project. A works alone for 3 days and the rest is done by B. The number of days required by B would be the same as the number of days for A to finish the whole project alone. How many days would A and B take to finish the project alone?

20. Place A and Place B are 36 km apart. P started from A and Q started from B, walking towards each other at the same time. After meeting, P took 2 hours 30 minutes to reach place B and Q took 1 hour and 36 minutes to reach place A. Find their speed.

(This chapter is translated to English by courtesy of Ms. YIK Kwan Ying, and reviewed by courtesy of Mr. SIN Wing Sang , Edward.)