Chapter 7 Factorization

7.1 Factorization

While learning fractions in arithmetic, we often need to simplify a fraction or find the common denominator of two fractions. Hence we have to factorize a number into prime numbers. e.g. decompose 33 into 3×11, and 42 into $2\times3\times7$.

While learning algebraic fractions in algebra we often simplify an algebraic fraction or find the common denominator of several fractions. Hence we need to express a polynomial as a product of integral algebraic expressions.

Expressing a polynomial as a product of integral algebraic expressions is called **factorization**.

Factorization is just the reverse of multiplication. For example, in expanding $(a+b)(a-b)$ we get a^2-b^2. Turning the process in reverse, transforming a^2-b^2 into $(a+b)(a-b)$ is factorization. Hence by studying the multiplication of some integral algebraic expressions, we can develop methods for factorization.

The followings are some common methods of factorization.

7.2 Taking common factors method

Let's look at an example of how to factorize a polynomial $am+bm-cm$. From multiplication, we get

$$m(a+b-c)=ma+mb-mc,$$

this means that $ma+mb-mc$ can be decomposed into the products of two factors m and $a+b-c$.

In the equation $ma+mb-mc=m(a+b-c)$, on the left hand side is the polynomial $ma+mb-mc$. There is a factor m common to all the terms of the polynomial. We call this factor **a common factor of the terms of the polynomial**. So m is the common factor of all the terms of the polynomial $ma+mb-mc$.

On the right hand side, the expression is the result of the polynomial after factorization. It is the product of the two factors, m and $a+b-c$. m is the common factor of $ma+mb-mc$. $a+b-c$ is the quotient obtained from dividing $ma+mb-mc$ by m.

From the above example, we can see that: If there is a common factor in the polynomial, we can take the common factor out, use it to divide the polynomial to find the quotient as the other factor. Then express the polynomial as the product of the two factors. This method of factorization is called **taking common factors method**.

【Example 1】 Factorize $4a^3b^2-6ab^3c$.

Solution $4a^3b^2-6ab^3c=2ab^2\bullet 2a^2-2ab^2\bullet 3bc$.

$$=2ab^2(a^2-3bc)$$

From example 1, it can be seen that the common factor is the product of the highest common factor of the coefficients multiplied by all the letters each of power with the lowest index.

【Example 2】 Factorize $3x^2-6xy+x$.

Solution $3x^2-6xy+x=x\bullet 3x-x\bullet 6y+x\bullet 1$.

$$=x(3x-6y+1)$$

NOTE: In example 2, $x=x\bullet 1$. In writing a term, the coefficient of '1' is usually omitted. However in factorization, care must be taken to note that the coefficient of '1' is there after the trailing algebraic letter is taking away as common factor.

【Example 3】 Factorize $-4m^3+16m^2-6m$.

Solution $-4m^3+16m^2-6m=-(4m^3-16m^2+6m)$.

$$=-(2m\bullet 2m^2-2m\bullet 8m+2m\bullet 3)$$

$$=-2m(2m^2-8m+3)$$

If the coefficient of the first term of the polynomial is negative, then take the negative sign out so that the coefficient of the first term becomes positive. When the negative sign is taken outside the bracket, the signs of all the terms inside the bracket have to be changed accordingly.

Practice

1. (*Mental*) Which of the following change from left side to the right side of the equation is factorization? Which is not? Why?
   (1) $(x+2)(x-3)=x^2-x-6$; (2) $x^2-4=(x+2)(x-2)$;
   (3) $x^2-4+3x=(x+2)(x-2)+3x$.
2. (*Mental*) If we use the method of taking common factor method, which factor should be taken out?
   (1) $ax+ay$; (2) $3mx-6nx$;
   (3) $4a^2+10ab$; (4) $15a^2+5a$;
   (5) x^2y+xy^2; (6) $12xyz-9x^2y^2$.
3. Factorize the following expressions:
   (1) $nx-ny$; (2) a^2+ab;
   (3) $4x^3-6x^2$; (4) $8m^2n+2mn$;
   (5) $3a^2y-3ay+6y$; (6) $a^2b+5ab-b$;
   (7) $-x^2+xy-xz$; (8) $-24x^2y-12xy^2+28y^3$;
   (9) $-3ma^2-6ma^2+12ma$; (10) $56x^3yz+14x^2y^2z-21xy^2z^2$

【Example 4】 Factorize $2a(b+c)-3(b+c)$.

Analysis: Regard this polynomial as the composite of two expressions: The first expression is $2a(b+c)$ and the second expression is $-3(b+c)$. The two expressions have the common factor $b+c$. So the method of taking common factors can be used.

Solution $2a(b+c)-3(b+c)=(b+c)(2a-3)$.

【Example 5】 Factorize $6(x-2)+x(2-x)$.

Analysis: Regard this polynomial as the composite of two expressions. Since $2-x=-(x-2)$, so each term has common factor $x-2$, and the method of taking common factors can be used.

Solution $6(x-2)+x(2-x)=6(x-2)-x(x-2)$
$=(x-2)(6-x)$

【Example 6】 Factorize $5(x-y)^3+10(y-x)^2$.

Analysis: This polynomial can be regarded as the composite of two expressions too. Since $(y-x)^2=[-(x-y)]^2=(x-y)^2$, so each term has the common factor $(x-y)^2$, and the method of taking common factors can be used.

Solution $5(x-y)^3+10(y-x)^2=5(x-y)^3+10(x-y)^2$
$=5(x-y)^2\bullet(x-y)+5(x-y)^2\bullet 2$
$=5(x-y)^2[(x-y)+2]$
$=5(x-y)^2(x-y+2)$

【Example 7】 Factorize $18b(a-b)^2-12(a-b)^3$.

Solution $18b(a-b)^2-12(a-b)^3=6(a-b)^2[3b-2(a-b)]$
$=6(a-b)^2(3b-2a+2b)$
$=6(a-b)^2(5b-2a)$

Practice

1. Add the correct sign (positive or negative) to the front of the brackets on the right so that the left hand side is equal to the right hand side:
   (1) $y-x=\quad(x-y)$; (2) $b-a=\quad(a-b)$;
   (3) $d+c=\quad(c+d)$; (4) $-z-y=\quad(y+z)$;
   (5) $(b-a)^2=\quad(a-b)^2$; (6) $-x^2+y^2=\quad(x^2-y^2)$;
   (7) $(x-y)^3=\quad(y-x)^3$;
   (8) $(1-x)(x-2)=\quad(x-1)(x-2)$.

Practice

2. Factorize the following expression:

(1) $a(x+y)+b(x+y)$; (2) $6(p+q)^2-2(p+q)$;

(3) $2(x-y)^2-x(x-y)$; (4) $m(a-b)-n(b-a)$;

(5) $3(y-x)^2+2(x-y)$; (6) $m(m-n)^2-n(n-m)^2$;

(7) $mn(m-n)-m(n-m)^2$; (8) $2x(x+y)^2-(x+y)^3$.

Exercise 21

1. Factorize the following expression:

(1) $cx-cy+cz$; (2) $px-qx-rx$;

(3) $15a^3-10a^2$; (4) $12abc-2bc^2$;

(5) $4x^2y-xy^2$; (6) $63pq+14pq^2$;

(7) $24a^3m-18a^2m^2$; (8) x^ny-x^nz.

2. Fill in the bracket with the correct polynomial so that the right hand side is equal to the left hand side:

(1) $14abx-8ab^2x+2ax=2ax($ $)$;

(2) $-7ab-14abx+49aby=-7ab($ $)$.

3. Factorize the following expression:

(1) $15x^3y^2+5x^2y-20x^2y^3$;

(2) $6m^2n-15mn^2+30m^2n^2$;

(3) $-16x^4-32x^3+56x^2$;

(4) $-4a^3b^2-6a^2b-2ab$.

4. Factorize the following expression:

(1) $x(a+b)-y(a+b)$;

(2) $5x(x-y)+2y(x-y)$;

(3) $6q(p+q)-4p(p+q)$;

(4) $(m+n)(p+q)-(m+n)(p-q)$;

(5) $a(a-b)+(a-b)^2$;

(6) $x(x-y)^2-y(x-y)$;

(7) $(2a+b)(2a-3b)-3a(2a+b)$;

(8) $x(x+y)(x-y)-x(x+y)^2$.

5. Factorize the following expression:

(1) $p(x-y)-q(y-x)$;

(2) $m(a-m)+2(3-a)$;

(3) $(a+b)(a-b)-(b+a)$;

(4) $a(x-a)+b(a-x)-c(x-a)$;

(5) $10a(x-y)^2-5b(y-x)$;

(6) $x(x-y)^2-y(y-x)$;

(7) $3(x-1)^3y-(1-x)^3z$;

(8) $x(a-x)(a-y)-y(x-a)(y-a)$.

6. Use factorization to compute the following:

(1) $21\times3.14+62\times3.14+17\times3.14$;

(2) $2.186\times1.237-1.237\times1.186$.

7. Given the formula $V=IR_1+IR_2+IR_3$. When $I=2.5$, $R_1=19.7$, $R_2=32.4$, $R_3=35.9$, use factorization to simplify the expression and then find the value of *V*.

7.3 Formula Method

As factorization means the reverse of multiplication, we can regard the reverse of some multiplication formula as a method to factorize certain polynomials. This method is called the Formula Method.

1. Difference of Two Squares Method

We know that

$$(a+b)(a-b)=a^2-b^2$$

In the reverse, we get

$$\boxed{\mathbf{a^2-b^2=(a+b)(a-b)}}$$

That is, **the difference of two squares** is equal to the product of the sum and the difference of the two numbers. Using this formula, we can factorize an expression after formulating it into the difference of two squares.

For example, to factorize x^2-16 and $9m^2-4n^2$, we cannot use taking common factors. But we can see that $16=4^2$, $9m^2=(3m)^2$, $4n^2=(2n)^2$, so $x^2-16=x^2-4^2$ and $9m^2-4n^2=(3m)^2-(2n)^2$. Since the polynomials can be fouulated into the difference of two squares, we can use the formula $a^2-b^2=(a+b)(a-b)$ to factorize the polynomials.

$$x^2-16=x^2-4^2=(x+4)(x-4)$$

$$\updownarrow$$

$$a^2-b^2=(a+b)(a-b)$$

$$9m^2-4n^2=(3m)^2-(2n)^2=(3m+2n)(3m-2n)$$

$$\updownarrow$$

$$a^2-b^2=(a+b)(a-b)$$

【Example 1】 Factorize the following expressions.:

(1) $1-25b^2$;

(2) $x^2y^2-z^2$;

(3) $\frac{4}{9}m^2-0.01n^2$.

Solution (1) $1-25b^2=1-(5b)^2=(1+5b)(1-5b)$;

(2) $x^2y^2-z^2=(xy)^2-z^2=(xy+z)(xy-z)$;

(3) $\frac{4}{9}m^2-0.01n^2=\left(\frac{2}{3}m\right)^2-(0.1n)^2$

$$=\left(\frac{2}{3}m+0.1n\right)\left(\frac{2}{3}m-0.1n\right)$$

【Example 2】 Factorize the following expressions:

(1) $(x+p)^2-(x+q)^2$;

(2) $16(a-b)^2-9(a+b)^2$.

Analysis: In the first question, $(x+p)^2-(x+q)^2$ is the difference of the two squares, $x+p$ and $x+q$. In the second question, $16(a-b)^2-9(a+b)^2=[4(a-b)]^2-[3(a+b)]^2$, so it is the difference of the squares of $4(a-b)$ and of $3(a+b)$. Thus both can be factorized by Formula Method.

Solution (1) $(x+p)^2-(x+q)^2=[(x+p)+(x+q)][(x+p)-(x+q)]$

$$=(2x+p+q)(p-q)$$

(2) $16(a-b)^2-9(a+b)^2$

$$=[4(a-b)]^2-[3(a+b)]^2$$

$$=[4(a-b)+3(a+b)][4(a-b)-3(a+b)]$$

$$=(7a-b)(a-7b)$$

【Example 3】 Factorize the following expression:

(1) x^5-x^3; (2) x^4-y^4.

Solution (1) $x^5-x^3=x^3(x^2-1)=x^3(x+1)(x-1)$;

(2) $x^4-y^4=(x^2)^2-(y^2)^2$

$$=(x^2+y^2)(x^2-y^2)$$

$$=(x^2+y^2)(x+y)(x-y)$$

NOTE: (1) If each of the terms of the polynomial has a common factor, take that common factor out first and continue to factorize the remaining expression.;
(2) In factorization, always factorize the expression until no more factorization can be done to the expression.

Practice

1. Fill in the bracket with the correct positive expression so that both sides of the equation are equal:
   (1) $4x^2 = (\quad)^2$; (2) $25m^2 = (\quad)^2$;
   (3) $36a^4 = (\quad)^2$; (4) $0.09b^2 = (\quad)^2$;
   (5) $81n^6 = (\quad)^2$; (6) $\frac{16}{49}c^2 = (\quad)^2$;
   (7) $64x^2y^2 = (\quad)^2$; (8) $100p^4q^2 = (\quad)^2$.
2. (*Mental*) Factorize the following expression:
   (1) $x^2 - 4$; (2) $9 - y^2$;
   (3) $1 - a^2$; (4) $4x^2 - y^2$.
3. Factorize the following expression:
   (1) $a^2 - \frac{1}{9}x^2$; (2) $36 - m^2$;
   (3) $4x^2 - 9y^2$; (4) $0.81a^2 - 16b^2$;
   (5) $36n^2 - 1$; (6) $25p^2 - 49q^2$.
4. Can the following expression be factorized using the Difference of Two Squares formula? If yes, what are the factors? if not, state the reasons.
   (1) $x^2 + y^2$; (2) $x^2 - y^2$;
   (3) $-x^2 + y^2$; (4) $-x^2 - y^2$.
5. Factorize the following expression:
   (1) $4a^2 - (b+c)^2$; (2) $(3m+2n)^2 - (m-n)^2$;
   (3) $2ab^3 - 2ab$; (4) $x^3 - 16x$;
   (5) $1 - a^4$; (6) $-x^4 + 16$.

2. Complete Square Formula Method

We know that

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

Reversing both sides of the equation, we get

$$\boldsymbol{a^2 + 2ab + b^2 = (a+b)^2}$$
$$\boldsymbol{a^2 - 2ab + b^2 = (a-b)^2}$$

That is to say, the sum of the squares of two numbers, plus (or minus) twice the product of the two numbers, is equal to the square of the sum (or difference) of the two numbers. Therefore we call expressions of the form $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$ **complete square expressions**, and the above equations are called **Complete Square Identities**. Using these two identities, we can factorize complete square expressions accordingly.

Let us try some examples, factorize polynomial $x^2 + 6x + 9$ and polynomial $4x^2 - 20x + 25$. Polynomial $x^2 + 6x + 9$ has three terms, (i) the first term is square of *x*, (ii) the third term 9 is square of 3, (iii) the second term 6*x* is twice the product of 3 and *x*, therfore polynomial $x^2 + 6x + 9$ is a complete square expression. Using complete square identity $a^2 + 2ab + b^2 = (a+b)^2$ to factorize the expression, we get

$$x^2 + 6x + 9 = x^2 + 2 \bullet x \bullet 3 + 3^2 = (x+3)^2$$
$$\Updownarrow \qquad\qquad \Updownarrow \qquad\qquad \Updownarrow$$
$$a^2 + 2 \bullet a \bullet b + b^2 = (a+b)^2$$

In a similar manner, polynomial $4x^2 - 20x + 25$ also has 3 terms, the first term $4x^2$ is square of 2*x*, the third term 25 is suquare of 5, the second term $-20x$ is the opposite number to twice the product of 2*x* and 5, therefore polynomial $4x^2 - 20x + 25$ is a complete square expression. Using complete square identity $a^2 - 2ab + b^2 = (a-b)^2$ to factorize the expression, we get

$$4x^2-20x+25=(2x)^2-2\bullet 2x\bullet 5+\ 5^2=\ (2x-5)^2$$
$$a^2\ \ -2\bullet\ a\ \bullet b+b^2=\ (\ a-b)^2$$

【Example 4】 Factorize $25x^4+10x^2+1$.

Solution $25x^4+10x^2+1=(5x^2)^2+2\bullet 5x^2\bullet 1+1^2=(5x^2+1)^2$.

【Example 5】 Factorize $-x^2-4y^2+4xy$.

Solution $-x^2-4y^2+4xy=-(x^2+4y^2-4xy)$

$$=-(x^2-4xy+4y^2)$$
$$=-[x^2-2\bullet 2xy+(2y)^2]$$
$$=-(x-2y)^2$$

【Example 6】 Factorize $3ax^2+6axy+3ay^2$.

Solution $3ax^2+6axy+3ay^2=3a(x^2+2xy+y^2)=3a(x+y)^2$.

Practice

1. Factorize the following expression:
   (1) x^2+2x+1; (2) $4a^2+4a+1$;
   (3) $1-6y+9y^2$; (4) $1+m+\frac{m^2}{4}$.
2. Is the following expression a complete square expression? If yes, how can it be factorized? If not, state the reasons.
   (1) x^2-4x+4; (2) $1+16a^2$;
   (3) $4x^2+4x-1$; (4) x^2+xy+y^2.
3. Factorize the following expression:
   (1) $x^2-12xy+36y^2$; (2) $25p^2+10pq+q^2$;
   (3) $\frac{m^2}{9}+\frac{2mn}{3}+n^2$; (4) $a^2-14ab+49b^2$;
   (5) $16a^4+24a^2b^2+9b^4$; (6) $(x+y)^2-10(x+y)+25$;
   (7) $-2xy-x^2-y^2$; (8) $ax^2+2a^2x+a^3$.

3. Sum of Two Cubes formula and Difference of Two Cubes formula

We know that

$$(a+b)(a^2-ab+b^2)=a^3+b^3$$
$$(a-b)(a^2+ab+b^2)=a^3-b^3$$

Reversing the process we get

$$\boldsymbol{a^3+b^3=(a+b)(a^2-ab+b^2)}$$
$$\boldsymbol{a^3-b^3=(a-b)(a^2+ab+b^2)}$$

That is, **the sum (difference) of two cubes** is equal the product of the sum of the two numbers multiplied by the combined result of the sum of the squares of the two numbers minus (plus) the product of the two numbers.

NOTE: In the formula, a^2-ab+b^2 and a^2+ab+b^2 are not perfect squares.

For example, factorize x^3+8 and $27-8a^3$. Since $8=2^3$, so x^3+8 is in the form of the sum of two cubes. We can use the formula, $a^3+b^3=(a+b)(a^2-ab+b^2)$ to factorize the expression.

$$x^3+8=x^3+2^3\ =\ (x+2)(x^2-x\bullet 2+2^2)$$
$$a^3+b^3=(a+b)(a^2-a\bullet b+b^2)$$

Similarly since $27=3^3, 8a^3=(2a)^3$ and so $27-8a^3$ is in the form of the difference of two cubes. We can use the formula $a^3-b^3=(a-b)(a^2+ab+b^2)$ to factorize the expression.

$$27-8a^3=3^3-(2a)^3=\ (3-2a)[3^2+3\bullet 2a+(2a)^2]$$
$$a^3-\ \ b^3=(a-\ \ b)\,(a^2+a\bullet\ \ b+\ \ b^2)$$

【**Example 7**】Factorize the following expressions:

(1) $27-x^6$; (2) $1+\frac{a^3b^3}{8}$.

Solution (1) $27-x^6=3^3-(x^2)^3$

$$=(3-x^2)[3^2+3\bullet x^2+(x^2)^2]$$

$$=(3-x^2)(9+3x^2+x^4)$$

(2) $1+\frac{a^3b^3}{8}=1^3+\left(\frac{ab}{2}\right)^3$

$$=\left(1+\frac{ab}{2}\right)\left[1^2-1\bullet\frac{ab}{2}+\left(\frac{ab}{2}\right)^2\right]$$

$$=\left(1+\frac{ab}{2}\right)\left(1-\frac{ab}{2}+\frac{a^2b^2}{4}\right)$$

【**Example 8**】Factorize $x-xy^3$.

Solution $x-xy^3=x(1-y^3)=x(1-y)(1+y+y^2)$.

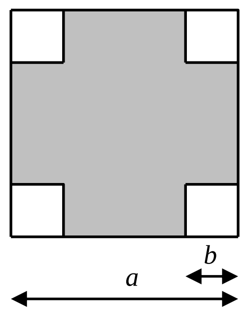
Practice

1. Fill in the bracket with the correct monomial so that both sides of the equation are equal:
   (1) $64a^3=(\quad)^3$; (2) $125n^6=(\quad)^3$;
   (3) $0.001x^3=(\quad)^3$; (4) $-27a^3b^3=(\quad)^3$.
2. Factorize the following expression:
   (1) a^3+1; (2) $1-m^3$;
   (3) $8p^3-q^3$; (4) $27+x^3$;
   (5) $1-27y^6$; (6) $125m^3+8n^3$;
   (7) p^6-64q^3; (8) $m^3n^3-\frac{1}{125}$.
3. Factorize the following expression:
   (1) $81+3x^3$; (2) y^4-8y;
   (3) $-27-a^3$; (4) $4m^4-\frac{m}{2}$.

Exercise 22

Factorize the following expressions (for questions 1 to 3):

1. (1) a^2-49; (2) $64-x^2$;
   (3) $1-36b^2$; (4) m^2-81n^2;
   (5) $0.49p^2-144q^2$; (6) $121x^2-4y^2$;
   (7) $a^2p^2-b^2q^2$; (8) $\frac{25}{4}a^2-x^2y^2$.
2. (1) $(m+n)^2-n^2$; (2) $169(a-b)^2-196(a+b)^2$;
   (3) $(2x+y)^2-(x+2y)^2$; (4) $(a+b+c)^2-(a+b-c)^2$;
   (5) $4(2p+3q)^2-(3p-q)^2$; (6) $(x^2+y^2)^2-x^2y^2$.
3. (1) $81a^4-b^4$; (2) $8y^4-2y^2$;
   (3) $3ax^2-3ay^4$; (4) m^4-1.
4. Use factorization to evaluate the following:
   (1) 758^2-258^2; (4) 429^2-171^2;
5. As shown in the figure, in a square cardboard with side a cm, four squares with side b $\left(b<\frac{a}{2}\right)$ cm are cut away from the four corners. Given that a = 13.2, and b = 3.4, use factorization method to find the remaining area.



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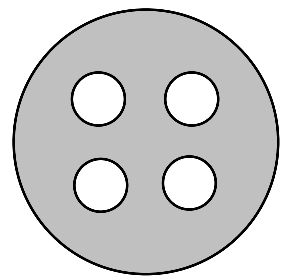

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6. In the figure, in a circular cardboard with radius R, four circles with radius r are cut away. Given that R = 7.8 cm and r = 1.1 cm, use factorization method to find the area of the remaining part (Take π as 3.14, give the answer correct to 2 significant numbers.).

Factorize the following expressions (for question 7 to 11):

7. (1) x^2-2x+1; (2) $a^2+8a+16$;
   (3) $1-4t+4t^2$; (4) $m^2-14m+49$;
   (5) $b^2-22h+121$; (6) $y^2+y+\frac{1}{4}$.

8. (1) $25m^2-80m+64$; (2) $4a^2+36a+81$;
   (3) $4p^2-20pq+25q^2$; (4) $\frac{x^2}{4}+xy+y^2$.

9. (1) $25a^4-40a^2b^2+16b^4$; (2) $36x^4-12x^2y+y^2$;
   (3) $a^2b^2-4ab+4$; (4) $16-8xy+x^2y^2$.

10. (1) $(x+y)^2+6(x+y)+9$; (2) $a^2-2a(b+c)+(b+c)^2$;
   (3) $4-12(x-y)+9(x-y)^2$; (4) $(m+n)^2+4m(m+n)+4m^2$.

11. (1) $2xy-x^2-y^2$; (2) $4xy^2-4x^2y-y^3$;
   (3) $3-6x+3x^2$; (4) $-a+2a^2-a^3$.

Factorize the following expressions (for questions 12 to 15):

12. (1) x^3-y^3; (2) x^3+125;
   (3) a^3+8b^3; (4) $27m^3-n^3$.

13. (1) $1-\frac{1}{8}a^3$; (2) $0.064p^3+1$;
   (3) x^3y^3-27; (4) p^3-q^6.

14. (1) $(a+b)^3+c^3$; (2) $(2x+1)^3-x^3$;
   (3) $-a-a^4$; (4) $3x^3+24$.

15. (1) $x^5-x^3y^2$; (2) $16x^5+8x^3y^2+xy^4$;
   (3) x^4+xy^3; (4) $16x^4-y^4$.

16. First factorize the two expressions and then point out the common factor of the two expressions:
   (1) x^2-4y^2 and $x^2+4xy+4y^2$;
   (2) $9x^2-24x+16$, $27x^3-64$ and $9x^2-16$.

7.4 Trinomial (polynomial with three terms) of Second Degree Method

We know

$$(x+a)(x+b)=x^2+(a+b)x+ab$$

Reversing the process, we get that

$$\boldsymbol{x^2+(a+b)x+ab=(x+a)(x+b)}$$

Thus, for a polynomial of 2nd degree and with three terms, if the constant term q can be decomposed into the product of two factors a, b and $a+b=p$ then the polynomial can be factorized

$$x^2+px+q=x^2+(a+b)x+ab=(x+a)(x+b).$$

Using this formula, certain polynomial which is of degree two, with three terms and the coefficient of the second degree term is 1, can be factorized.

For example, factorize the polynomial. It is of degree 2 and with 3 terms. We try to decompose 6 into the product of two factors and the sum of the two factors is equal to the coefficient of the 1st degree term,5. As $6=2\times3$ and $2+3=5$, so

$$x^2+5x+6=(x+2)(x+3).$$

【Example 1】 Factorize x^2+3x+2.

Analysis: The constant term 2 can be decomposed into the product of 1 and 2 or -1 and -2. Among them only the sum of 1 and 2 equals to 3, same as the coefficient of 1st degree term.

Solution Since $2 = 1\times 2$ and $1+2=3$,

$$x^2+3x+2=(x+1)(x+2).$$

【**Example 2**】Factorize x^2-7x+6.

Analysis: The constant term 6 can be decomposed into the product of 1 and 6 or -1 and -6 or 2 and 3 or -2 and -3. Among them only $(-1)+(-6)=-7$ which is equal to the coefficient of 1st degree term.

Solution Since $6=(-1)\times(-6)$ and $(-1)+(-6)=-7$,

$$x^2-7x+6=[x+(-1)][x+(-6)]=(x-1)(x-6).$$

【**Example 3**】Factorize $x^2-4x-21$.

Analysis: The constant term -21 can be the decomposed into the product of 1 and -21 or -1 and 21 or -3 and 7 or 3 and -7. Among them only $3+(-7)=-4$ which is equal to the coefficient of 1st degree term.

Solution Since $-21=3\times(-7)$ and $3+(-7)=-4$,

$$x^2-4x-21=(x+3)[x+(-7)]=(x+3)(x-7).$$

【**Example 4**】Factorize $x^2+2x-15$.

Solution Since $-15=(-3)\times 5$ and $(-3)+5=2$,

$$x^2+2x-15=[x+(-3)](x+5)=(x-3)(x+5).$$

From examples 1 to 4, it can be seen that: if the constant term is positive, it should be decomposed into two factors with the same sign. The sign should be the same as the coefficient of the 1st degree term. If the constant term is negative, it should be decomposed into two factors with different signs. The factor with greater absolute value should have the same sign with that of the coefficient of the 1st degree term.

【**Example 5**】Factorize the following expression:

(1) x^4+6x^2+8; (2) $(a+b)^2-4(a+b)+3$.

Solution (1) $x^4+6x^2+8=(x^2)^2+6(x^2)+8$

$$=[(x^2)+2][(x^2)+4]$$

$$=(x^2+2)(x^2+4)$$

(2) $(a+b)^2-4(a+b)+3=[(a+b)-1][(a+b)-3]$

$$=(a+b-1)(a+b-3)$$

【**Example 6**】Factorize $x^2-3xy+2y^2$.

Analysis: Take $x^2-3xy+2y^2$ as the polynomial of x with degree 2 and three terms. Then the constant term is $2y^2$. The coefficient of the 1st degree term is $-3y$. Decompose $2y^2$ into the product of $-y$ and $-2y$. $(-y)+(-2y)=-3y$ is just equal to the coefficient of the 1st degree term.

Solution $x^2-3xy+2y^2=(x-y)(x-2y)$.

【**Example 7**】Factorize $x^4-3x^3-28x^2$.

Solution $x^4-3x^3-28x^2=x^2(x^2-3x-28)=x^2(x+4)(x-7)$.

Practice

1. (*Mental*) Decompose the following number into the product of two factors (list out all the possible cases):
   (1) 3; (2) -5; (3) 12; (4) -8.

Factorize the following expressions (questions 2 to 5):

2. (1) x^2+4x+3; (2) $a^2+7a+10$;
   (3) $y^2-7y+12$; (4) q^2-6q+8;
   (5) x^2+x-20; (6) $m^2+7m-18$;
   (7) $p^2-5p-36$; (8) t^2-2t-8.

Practice

3.	(1)	$x^4 - x^2 - 20$;	(2)	$ax^2 + 7ax - 8$.
4.	(1)	$a^2 - 9ab + 14b^2$;	(2)	$x^2 + 11xy + 18y^2$.
5.	(1)	$x^2y^2 - 5x^2y - 6x^2$;	(2)	$-a^3 - 4a^2 + 12a$.

7.5 Grouping for Decompose Method

1. Grouping to Take Common Factor Method

Let's see how to factorize the polynomial

$$ax + ay + bx + by.$$

There is no common factor among the terms and no direct formula can be used. However there is a common factor a among the first two terms and a common factor b among the last two terms. So we can separately group the first two terms and the last two terms together. Then the polynomial becomes

$$(ax + ay) + (bx + by)$$

Taking the factors a and b from each group, we get

$$a(x + y) + b(x + y)$$

Now there is a common factor $x + y$ among the two groups. So we can take the factor $x + y$ out, and factorize the polynomial completely into $(x + y)(a + b)$. Thus

$$\begin{aligned} ax + ay + bx + by &= (ax + ay) + (bx + by) \\ &= a(x + y) + b(x + y) \\ &= (x + y)(a + b) \end{aligned}$$

Using grouping to factorize is called **grouping for decompose method**. From the above example, we can see that sometimes a common factor appears among the groups after some terms of a polynomial are grouped. If so, the polynomial can be factorized by grouping for decompose method.

【Example 1】 Factorize $a^2 - ab + ac - bc$.

Analysis: Group the first two terms and the last two terms separately into two groups. After taking the common factor a and c from each group respectively, there is a common factor $a - b$ in the each group, so $a - b$ can be taken .

Solution
$$\begin{aligned} a^2 - ab + ac - bc &= (a^2 - ab) + (ac - bc) \\ &= a(a - b) + c(a - b) \\ &= (a - b)(a + c) \end{aligned}$$

【Example 2】 Factorize $2ax - 10ay + 5by - bx$.

Analysis: The first two terms and the last two terms are grouped separately to form two groups. The terms in the group are arranged in descending power of x. Then the common factor $2a$ and $-b$ are taken out from the two groups respectively. Now $x - 5y$ is the common factor of the two groups. $x - 5y$ can be taken out as the factor of the whole expression.

Solution
$$\begin{aligned} 2ax - 10ay + 5by - bx &= (2ax - 10ay) + (5by - bx) \\ &= (2ax - 10ay) + (-bx + 5by) \\ &= 2a(x - 5y) - b(x - 5y) \\ &= (x - 5y)(2a - b) \end{aligned}$$

Line for thought: In example 1 and 2 are there any other ways of grouping? Is the result of factorization the same using other ways of grouping?

【Example 3】 Factorize $3ax + 4by + 4ay + 3bx$.

Analysis: In this polynomial, if the first two terms and the last two terms are grouped separately, it cannot be factorized. But if the first and the third terms are group together as well as the second and the fourth terms are grouped together, then a and b can be taken out as the common factor and $3x + 4y$ will be the other factor. So $3x + 4y$ can be taken out as the common factor of the whole expression.

Solution $3ax+4by+4ay+3bx=(3ax+4ay)+(4by+3bx)$

$$=(3ax+4ay)+(3bx+4by)$$
$$=a(3x+4y)+b(3x+4y)$$
$$=(3x+4y)(a+b)$$

【Example 4】 Factorize $m^2+5n-mn-5m$.

Solution $m^2+5n-mn-5m=(m^2-mn)+(5n-5m)$

$$=(m^2-mn)+(-5m+5n)$$
$$=m(m-n)-5(m-n)$$
$$=(m-n)(m-5)$$

Line for thought: In example 3 and 4 are there any other ways of grouping? Is the result of factorization the same using other ways of grouping?

Practice

Factorize the following expressions:

1. (1) $20(x+y)+x+y$; (2) $p-q+k(p-q)$;
   (3) $5m(a+b)-a-b$; (4) $2m-2n-4x(m-n)$.
2. (1) $ac+bc+2a+2b$; (2) $a^2+ab-ac-bc$;
   (3) $3a-ax-3b+bx$; (4) $xy-y^2-yz+xz$.
3. (1) $5ax+6by+5ay+6bx$; (2) $4x^2+3z-3xz-4x$.

2. Grouping to use Formula Method

【Example 5】 Factorize $x^2-y^2+ax+ay$.

Analysis: Group the first two terms together. Though there is no common factors among these two terms, they can be factorized using formula of the difference of squares. After that, $x+y$ is one of the factors. The third and the fourth terms can be grouped together and after taking out the common factor a, $x+y$ is the other factor. $x+y$ can be taken out as the common factor of the whole expression.

Solution $x^2-y^2+ax+ay=(x^2-y^2)+(ax+ay)$

$$=(x+y)(x-y)+a(x+y)$$
$$=(x+y)[(x-y)+a]$$
$$=(x+y)(x-y+a)$$

【Example 6】 Factorize $a^2-2ab+b^2-c^2$.

Analysis: Group the first three together to form a group. It is a perfect square and can be expressed as $(a-b)^2$. Now the whole expression can be factorized by using the formula of the difference of two squares.

Solution $a^2-2ab+b^2-c^2=(a^2-2ab+b^2)-c^2$

$$=(a-b)^2-c^2$$
$$=[(a-b)+c][(a-b)-c]$$
$$=(a-b+c)(a-b-c)$$

From example 5 and 6 it can be seen that if the terms of the polynomial are grouped in an appropriate way so that the terms in the groups can be factorized using formula. Then the polynomial can be factorized using grouping.

【Example 7】 Factorize $x^3+x^2y-xy^2-y^3$.

Solution $x^3+x^2y-xy^2-y^3=(x^3+x^2y)-(xy^2+y^3)$

$$=x^2(x+y)-y^2(x+y)$$
$$=(x+y)(x^2-y^2)$$
$$=(x+y)[(x+y)(x-y)]$$
$$=(x+y)^2(x-y)$$

NOTE: x^2-y^2 can be factorized and so it should be further factorized.

Now let's have a summary. A polynomial can be factorized with the following steps:

1. If there is a common factor among the terms, take the common factor out first;
2. If there is no common factor, see if a formula can be used;
3. If the above methods cannot be used, try to use the method of grouping for decompose method or the trinomial method to factorize the expression;
4. The factorization process must be pursued until each factor cannot be further factorized.

Practice

Factorize the following expression:

1. $4a^2 - b^2 + 6a - 3b$.
2. $9m^2 - 6m + 2n - n^2$.
3. $x^2 - y^2 - z^2 + 2yz$.
4. $x^3 - x^2y - xy^2 + y^3$.

7.6 Cross-Multiplication Method

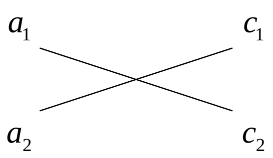
We know

$$(a_1x + c_1)(a_2x + c_2) = a_1a_2x^2 + a_1c_2x + a_2c_1x + c_1c_2$$
$$= a_1a_2x^2 + (a_1c_2 + a_2c_1)x + c_1c_2$$

Reversing the process, we get

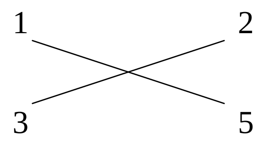
$$a_1a_2x^2 + (a_1c_2 + a_2c_1)x + c_1c_2 = (a_1x + c_1)(a_2x + c_2) .$$

Bearing in mind this equality, we can use it in the following ways to factorize some of the trinomials (polynomials with 3 terms are called trinomials) in 2nd degree such as $ax^2 + bx + c$. First express $a =$ a_1a_2 and $c = c_1c_2$, Write a_1 and a_2 , c_1 and c_2 as shown below:



Follow the straight lines to get the cross product of each line. Then the sum of the cross products is $a_1c_2 + a_2c_1$. If the sum is equal to the coefficient(b) of the x term of the polynomial, then the polynomial can be expressed as $(a_1x + c_1)(a_2x + c_2)$, where a_1 and c_1 are the number on the first row and a_2 and c_2 are the numbers on the second row.

For example, Factorize the trinomial $3x^2 + 11x + 10$ .We know $3 = 1 \times 3$ and $10 = 2 \times 5$. So expressing them on a cross as in the following diagram

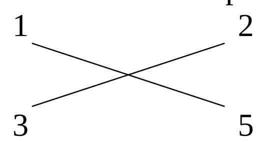


It can be seen that $1 \times 5 + 2 \times 3 = 11$, so

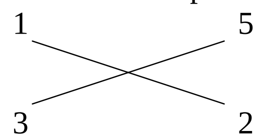
$$3x^2 + 11x + 10 = (x + 2)(3x + 5) .$$

This method of drawing a cross to help in factorizing a trinomial of 2nd degree is called **cross- multiplication method**.

NOTE: There are many cases for cross-multiplication of the factors. Very often many attempts are needed in order to decide whether it can be factorized and how it can be factorized. e.g. In the trinomial $3x^2 + 11x + 10$, the constant term can be written as the product of 1 and 10 , -1 and -10 , 2 and 5 or -2 and -5 . The cross-multiplication



is correct, while the cross-multiplication



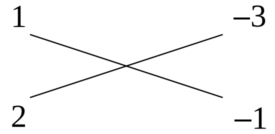
is wrong. So when using cross-multiplication method, many trials are needed .

【Example】 Factorize the following expression:

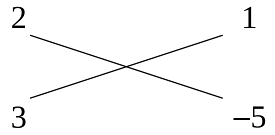
(1) $2x^2-7x+3$; (2) $6x^2-7x-5$;

(3) $5x^2+6xy-8y^2$.

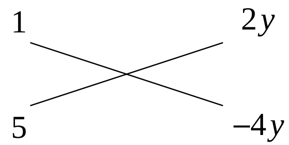
Solution (1) $2x^2-7x+3=(x-3)(2x-1)$



(2) $6x^2-7x-5=(2x+1)(3x-5)$



(3) $5x^2+6xy-8y^2=(x+2y)(5x-4y)$



Practice

Factorize the following expression:

1. (1) $2x^2+15x+7$; (2) $3a^2-8a+4$;
   (3) $8m^2+3m-5$; (4) $2x^2-7x-15$;
   (5) $5x^2+7x-6$; (4) $6y^2-11y-10$.
2. (1) $6a^2+17ab+12b^2$; (2) $15x^2-xy-6y^2$;
   (3) $5a^2b^2+23ab-10$; (4) $10x^2y^2-17abxy+3a^2b^2$.

Exercise 23

Factorize the following expressions (For questions 1 to 4):

1. (1) x^2+9x+8; (2) $x^2-10x+24$;
   (3) $x^2+3x-10$; (4) $x^2-3x-28$;
   (5) $a^2+4a-21$; (6) $m^2+4m-12$;
   (7) p^2-8p+7; (8) $b^2+11b+28$.

2. (1) x^4+7x^2-18; (2) x^6+8x^3+15;
   (3) $m^2x^2-8mx+12$; (4) $x^2y^2-7xy+10$.

3. (1) $x^2-7xy+12y^2$; (2) $a^2+2ab-15b^2$;
   (3) $m^2+4mn-12n^2$; (4) $p^2+9pq+18q^2$.

4. (1) $-x^2y+6xy-8y$; (2) $(m+n)^2-(m+n)-30$;
   (3) $ab^2+4abc+3ac^2$; (4) $(x-y)^2-3(x-y)-40$.

5. Factorize the following expressions and then point out the common factor in the three expressions:
   (1) $x^2+9x+14$, x^3-49x and $x^2+2x-35$;
   (2) $x^2+2x-63$, $x^2+18x+81$ and $x^2+12x+27$

Factorize the following expressions (For question 6 to 14):

6. (1) $am+an+bm+bn$; (2) $xy-xz+y-z$;
   (3) $a^2+ab+ac+bc$; (4) $ax-2bx+ay-2by$;
   (5) $4xy-3xz+8y-6z$; (6) x^3+3x^2+3x+9.

7. (1) $3xy-2x-12y+8$; (2) $ab-5bc-2a^2+10ac$;
   (3) $5ax+7xy-5bx-7by$; (4) $x^3y+3x-2x^2y^2-6y$.

8. (1) $6ax+15b^2y^2-6b^2x-15ay^2$; (2) $7x^2-3y+xy-21x$;
   (3) $3a^2+bc-3ac-ab$; (4) $a^2m+bn-an-abm$.

9. (1) $x^2-a^2-2x-2a$; (2) a^3-b^3-a+b;
   (3) $4x^2-4xy+y^2-a^2$; (4) $1-m^2-n^2+2mn$.

10. (1) $a-a^3$; (2) x^3-15x^2-16x;
(3) x^3y-xy^3; (4) $5x^5-15x^3y-20xy^2$;
(5) $x+x^4$; (6) a^4b-ab^4.

11. (1) $(x^2+3x)^2-(2x+6)^2$; (2) $1-26a^2+25a^4$;
(3) $(x^2+2x)^2-7(x^2+2x)-8$; (4) a^6+7a^3-8.

12. (1) $4x^2-y^2+2x-y$; (2) $(x+y)^4+(x+y)^2-20$;
(3) a^4+a^3+a+1; (4) $x^4y+2x^3y^2-x^2y-2xy^2$

13. (1) $2x^2+3x+1$; (2) $2y^2+y-6$;
(3) $6x^2-13x+6$; (4) $3a^2-7a-6$;
(5) $4n^2+4n-15$; (6) $6l^2+l-35$.

14. (1) $6x^2-11xy+3y^2$; (2) $4m^2+8mn+3n^2$;
(3) $16x^2-31xy-2y^2$; (4) $8m^2-22mn+15n^2$.

Chapter Summary

I. The main objective of this chapter is to teach the concept of factorization and to teach some methods to factorize a polynomial.

II. In learning to factorize a polynomial, we need to understand the relationship between factorization and integral multiplication. Integral multiplication is to multiply the factors of integral expressions together, which will result in a polynomial. Factorization is to decompose a polynomial as the product of factors, each of which is an integral expression. e.g. Expressing $(a+b)(a-b)$ as a^2-b^2 is integral multiplication and expressing a^2-b^2 as $(a+b)(a-b)$ is factorization.

$$(a+b)(a-b) \underset{\text{Factorization}}{\overset{\text{integral multiplication}}{\rightleftarrows}} a^2-b^2$$

The decision on when to use integral multiplication and when to use factorization, will depend on what is needed in the circumstances.

III. Factorization methods taught in this chapter are as follows:

(i) The Taking Common Factor Method is the most basic method used in factorization. As long as there are common factors among the terms, take the common factors out first.

(ii) The Formula Method is a very powerful method in factorization. To apply the formula method, it is important to know the special forms and their characteristics. In this chapter, we have learnt the following five formula or identities:

$$a^2-b^2=(a+b)(a-b)$$
$$a^2\pm 2ab+b^2=(a\pm b)^2$$
$$a^3\pm b^3=(a\pm b)(a^2\mp ab+b^2)$$

(iii) The Cross-Multplication Method can be used to solve some trinomials in 2nd degree. If the trinomial can be expressed in the form of $x^2+(a+b)x+ab$, then it can be factorized as $(x+a)(x+b)$. The key is to know the relationship of *a* and *b* with the constant term and with the coefficient of the 1st degree term. Then with due experience and intelligence, we can decompose the constant term into multiplicative factors *a* and *b* in such a way that the sum of *a* and *b* is equal to the coefficient of the 1st degree term.

(iv) To apply the Grouping for Decompose Method well, the key for success is the ability to foresee the possibilities of the next step. We have learnt in some examples that the next step may be the Taking Common Factor Method or the Formula Method.

Since there are many methods of factorization, we have to analyze the situation on-hand to be able to apply the most appropriate method flexibly to tackle the problem.

Revision Exercise 7

Factorize the following expressions (For question1 to 4):

1. (1) x^2-64; (2) x^3-64;
   (3) x^4+64x; (4) x^4-64x^2;
   (5) $a^{m+1}+a^m$; (6) $y^{n+2}-y^n$;
   (7) $(a-b)(x-y)-(b-a)(x+y)$;
   (8) $x(p-q)-y(p-q)+z(q-p)$;
   (9) $25(x+y)^2-16(x-y)^2$;
   (10) $p^2(p+q)^2-q^2(p-q)^2$;
   (11) $(a+b+c)^2-(a-b-c)^2$;
   (12) $(a+b)^2+2(a+b)-15$;

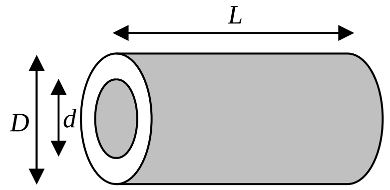
2. (1) m^3n^3+27; (2) $1-\frac{1}{64}a^3$;
   (3) $4-(3a+2b)^2$; (4) $(x^2+4)^2-16x^2$;
   (5) $x^2-xy-30y^2$; (6) $a^2x^2+16ax+64$;
   (7) $a^4-5a^2b^2+4b^4$; (8) $x^5-x^3y^2-12xy^4$;
   (9) $(a-b)^{n+2}-(a-b)^n$; (10) $x^{n+1}-3x^n+2x^{n-1}$;
   (11) $(x+y)^2-14y(x+y)+49y^2$;
   (12) $(3a-4b)(7a-8b)+(11a-12b)(7a-8b)$.

3. (1) $x^3z-4x^2yz+4xy^2z$; (2) $(x+2)(x+3)+x^2-4$;
   (3) x^2-4y^2+x+2y; (4) $x^2-6x+9-y^2$;
   (5) $a^3x^2-c^3x^2-a^3y^2+c^3y^2$; (6) $(a^2+b^2-1)^2-4a^2b^2$;
   (7) $x^5-x^3+x^2-1$;
   (8) $10a^2x+21xy^2-14ax^2-15ay^2$;
   (9) $\left(\frac{1}{2}x+\frac{2}{3}y-\frac{3}{4}z\right)^2-\left(\frac{1}{2}x-\frac{2}{3}y+\frac{3}{4}z\right)^2$;
   (10) $4b^2c^2-(b^2+c^2-a^2)^2$.

4. (1) $(ax+by)^2+(bx-ay)^2$; (2) $ab(c^2+d^2)+cd(a^2+b^2)$.

5. Use factorize to simplify the computation of the following:
   (1) 5% of 1297 minus 5% of 897. What is the difference?
   (2) 36% of 869 plus 54% of 869. What is the sum?

6. In a city the sewage system is to be repaired. A hollow concrete pipe with the following requirement is to be set up. Refer to the Diagram internal diameter d = 45 cm, external diameter D = 75 cm and length L = 300 cm. Using factorization method, find the volume (in m^3) of the concrete needed to make such a pipe (take π as 3.14 and give the answer in 2 significant figures.).



(No. 6)

7. Given two consecutive odd numbers $2n-1$ and $2n+1$ (where n is an integer), show that the difference of the squares of the two consecutive numbers (the greater one minus the smaller one, same in question 8) is a multiple of 8.

8. (1) Use algebraic expression to represent the difference of two consecutive numbers and the sum of the two consecutive numbers. Are these two expressions the same?
   (2) Use algebraic expression to represent the difference of two consecutive odd numbers and the sum of the two consecutive odd numbers. What is the relationship between these two expressions?

Factorize the following expressions (For questions 9 to10)

9. (1) $3x^2+x-2$; (2) $20y^2+y-1$;
   (3) $14x^2+x-3$; (4) $12t^2-8t-7$.

10. (1) $7p^2-5pq-2q^2$; (2) $6x^2-5xy-6y^2$;
    (3) $30a^2-ab-b^2$; (4) $18x^2-21xy+5y^2$.

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