International Young Mathematicians' Convention (IYMC)
2012 Team Contest –Junior level

1. A simple tune consists of the following 12 notes in the order:

C, E, E, E, G, G, D, F, F, A, B, B

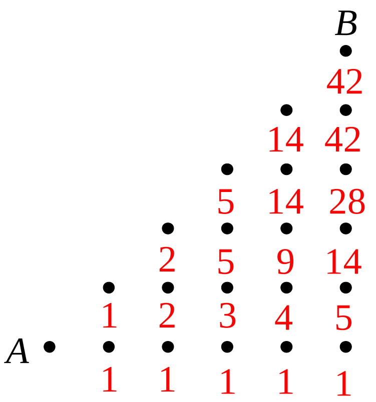
How many different tunes can be made with the same 12 notes?

【Solution】

There are 1 A, 2 Bs, 1 C, 1 D, 3 Es, 2 Fs and 2 Gs. So we can make $\frac{12!}{3!\,2!\,2!\,2!}=\frac{11!}{4}$ =9979200 different tunes.

Answer: 9979200

2. Any two adjacent dots in the diagram are 1 unit from each other. A path consists of horizontal and vertical segments between the dots joined end to end. How many paths from point A to point B are there with length 10 units?



【Solution】

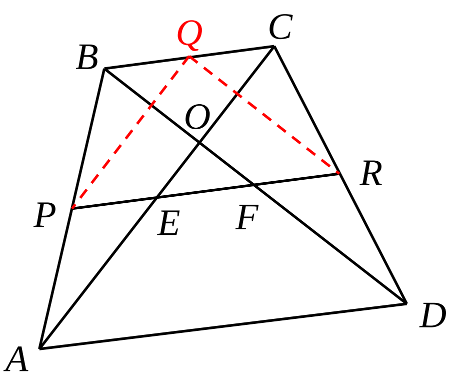
Since each path is formed by horizontal and vertical segments between the dots, the number of paths from *A* to the dot on the line *AB* is equal to the number of paths from *A* to the dot below it.

Observe that if you want to go from *A* to a dot, you have to pass through the left dot or the dot below.

On the figure, the number on the dot shows the number of paths from *A* to it, hence the answer is 42.

Answer: 42

3. In the figure, *ABCD* is a quadrilateral. If $AP=BP$, $CR=DR$ and $\angle OEF=\angle OFE$, prove that $AC=BD$.



【Solution】

Let *Q* be the midpoint of *BC*. Then the segment *PQ* is parallel to *AC* and equals half of *AC* , the segment *QR* is parallel to *BD* and equals half of *BD*.

Since $\angle QPR=\angle OEF=\angle OFE=\angle QRP$, hence $PQ=QR$, we can get $AC=2\cdot PQ=2\cdot QR=BD$.

4. How many different ordered triples (a,b,c) of positive integers satisfy

$$\left(\frac{a}{c}+\frac{a}{b}+1\right)\div\left(\frac{b}{a}+\frac{b}{c}+1\right)=11 \text{ and } a+2b+c\le 50?$$

【Solution】

Simplify the given expression as follows:

$$\left(\frac{ab+ac+bc}{bc}\right)\div\left(\frac{bc+ab+ac}{ac}\right)=11 \quad\Rightarrow \frac{ab+ac+bc}{bc}\cdot\frac{ac}{bc+ab+ac}=11$$

$$\Rightarrow \frac{a}{b}=11,\quad c\ne 0$$

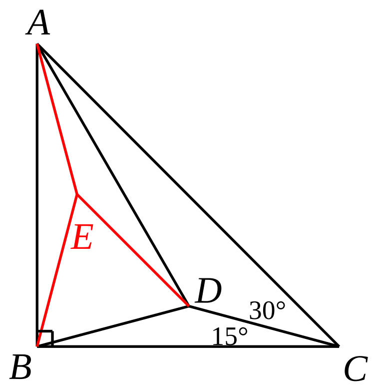
$$\Rightarrow a=11b$$

By substitution, the condition $a+2b+c\le 50$ becomes $13b+c\le 50$.
Since b and c are positive integers, then b can only take on the values 1, 2 or 3. The values of a correspond directly to the values of b, since $a=11b$.
When b = 3, there is 11 possible value of c. When b = 2, there are 24 possible values of c. When b = 1, there are 37 possible values of c. Therefore, the number of different ordered triples satisfying the given conditions is 11+ 24 + 37 = 72.

Answer: 72

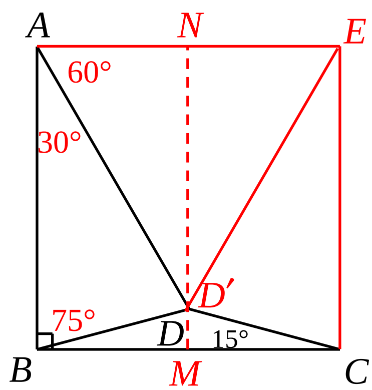
5. In the figure, $AB=BC$ and $\angle B = 90°$. If D is a point inside $\triangle ABC$ such that $BD= CD$ and $\angle ACD = 30°$. What is the measure of $\angle ADB$, in degree?

【Solution 1】



We know that $\angle DCB = \angle DBC = 15°$
Draw line segments AE and BE with equal lengths and let $\angle EAB = \angle EBA = 15°$. Since $AB=BC$, $AE=BE$, $BD=DC$ and $\angle EAB = \angle EBA = \angle DCB = \angle DBC = 15°$, we get triangles ABE and CBD are congruent, hence $EB=DB$.
Since $\angle EBD = 90° - 15° - 15° = 60°$, then BED is an equilateral triangle. Since $\angle AEB = 150°$, then $\angle AED = 360° - 150° - 60° = 150°$. We get $\angle EAD = \angle ADE = 15°$, hence $\angle ADB = 60° + 15° = 75°$.

【Solution 2】



Find point E such that $ABCE$ is a square. Let M and N be the midpoint of BC and AE, respectively. Thus MN is the perpendicular bisector of BC and hence D is on MN.
We can find a point D' on MN such that $AD'=AB$.
Then $ED'=AD'=AB=AE$, hence $AD'E$ is an equilateral triangle. So $\angle BAD'=90°-60°=30°$, and we can get

$$\angle ABD'=\angle AD'B=\frac{180°-30°}{2}=75°.$$

Hence $\angle D'BC=90°-\angle ABD'=15°$. Thus D and D' are coincide and hence $\angle ADB=\angle AD'B=75°$

Answer: 75°

6. If $\begin{cases} a+b+c=7 \\ a^2+b^2+c^2=21 \\ a^3+b^3+c^3=73 \end{cases}$, what is the value of $a^4+b^4+c^4$?

【Solution 1】

If cubic equation $x^3 - Ax^2 + Bx - C = 0$ has roots a, b, and c, when we expanding $(x-a)(x-b)(x-c)=0$, then we have

$$A = a+b+c$$
$$B = ab+bc+ca$$
$$C = abc$$

Where $B = ab+bc+ca = \frac{1}{2}((a+b+c)^2 - (a^2+b^2+c^2)) = \frac{49-21}{2} = 14$.

Since a, b, and c are roots of $x^3 - 7x^2 + 14x - C = 0$, and we have

$$a^3 - 7a^2 + 14a - C = 0$$
$$b^3 - 7b^2 + 14b - C = 0$$
$$c^3 - 7c^2 + 14c - C = 0$$

Adding, we get

$$(a^3+b^3+c^3) - 7(a^2+b^2+c^2) + 14(a+b+c) - 3C = 73 - 7\times 21 + 14\times 7 - 3C = 0.$$

Hence $C = 8$, and we get $x^3 - 7x^2 + 28x - 8 = 0$.

Multiplying the polynomial by x, we have $x^4 - 7x^3 + 14x^2 - 8x = 0$. Then

$$a^4 - 7a^3 + 14a^2 - 8a = 0$$
$$b^4 - 7b^3 + 14b^2 - 8b = 0$$
$$c^4 - 7c^3 + 14c^2 - 8c = 0$$

Adding, we get $(a^4+b^4+c^4) - 7(a^3+b^3+c^3) + 14(a^2+b^2+c^2) - 8(a+b+c) = 0$.

Hence $a^4+b^4+c^4 = 7\times 73 - 14\times 21 + 8\times 7 = 273$.

【Solution 2】

$$49 = (a+b+c)^2 = a^2+b^2+c^2 + 2(ab+bc+ca) = 21 + 2(ab+bc+ca)$$

Hence $ab+bc+ca = 14$.

$$343 = (a+b+c)^3 = a^3+b^3+c^3 + 3(a+b+c)(ab+bc+ca) - 3abc = 73 + 21\times 14 - 3abc$$

Hence $abc = 8$.

We can get

$$\begin{aligned} a^4+b^4+c^4 &= (a^2+b^2+c^2)^2 - 2(a^2b^2+b^2c^2+c^2a^2) \\ &= (a^2+b^2+c^2)^2 - 2((ab+bc+ca)^2 - 2abc(a+b+c)) \\ &= 21^2 - 2(14^2 - 2\times 8\times 7) \\ &= 273 \end{aligned}$$

Answer: 273