International Young Mathematicians' Convention (IYMC) 2012 Individual Contest –Senior level

1. If $15xy^2$ and $21xy$ are perfect squares, where x and y are positive integers, what is the smallest value of $x + y$?

【Solution】

Since $15xy^2$ and $21xy$ are perfect squares, the prime factors of each term must occur an even number of times.

Consider $15xy^2 = 3\cdot 5\cdot xy^2$, where the factor of y appears twice, then $3\cdot 5\cdot x$ must be a square. The smallest value of x for which $3\cdot 5\cdot x$ is a perfect square is $3\cdot 5 = 15$.

Now consider $21xy = 3\cdot 7\cdot (3\cdot 5)\cdot y$, it follows the smallest value of y such that $21xy$ is a perfect square is $5\cdot 7 = 35$.

Therefore, the smallest value of $x + y$ is $15 + 35 = 50$. ***Answer:*** 50

2. Determine $y - x$ if x and y are real numbers that satisfy $2^x - 2^y = 1$ and $4^x - 4^y = \frac{5}{3}$.

【Solution】

Note that $\frac{5}{3} = 4^x - 4^y = (2^x + 2^y)(2^x - 2^y) = 2^x + 2^y$. Therefore,

$$2^x = \frac{(2^x + 2^y) + (2^x - 2^y)}{2} = \frac{\frac{5}{3} + 1}{2} = \frac{4}{3} \quad\text{and}\quad 2^y = \frac{(2^x + 2^y) - (2^x - 2^y)}{2} = \frac{\frac{5}{3} - 1}{2} = \frac{1}{3},$$

which implies $2^{y-x} = \frac{2^y}{2^x} = \frac{1}{4} = 2^{-2}$. Thus $y - x = -2$. ***Answer:*** -2

3. A password consists of four distinct digits such that their sum is 19 and such that exactly two of these digits are primes. For example 0397 is a possible password. How many possible passwords are there?

【Solution】

The singles digit primes are 2, 3, 5, 7 from which we must choose two. The other two digits must be chosen from 0, 1, 4, 6, 8, 9 so that the sum of four digits is 19.

Ignoring order, the possible choices for the four digits are :

Primes	Sum of others' digits	Others
2 and 3	14	6 and 8;
2 and 5	12	4 and 8;
2 and 7	10	1 and 9; 4 and 6;
3 and 5	11	None
3 and 7	9	0 and 9; 1 and 8;
5 and 7	7	1 and 6;

This gives 7 choices for the four digits, and each choice can be arranged in 4×3×2×1=24 different ways, making a total of 7×24=168 passwords.

Answer: 168

4. For any positive integer *n*, we define *n*! as the product of the integers from 1 to *n*, and call it the factorial of *n*. Also 0! is defined as 1. Some numbers are equal to the sum of the factorials of their digits. For example 40585=4!+0!+5!+8!+5!. Find such a number with three digits.

【Solution】

Call the three-digit number *A*. Write down the factorials : 0!=1, 1!=1, 2!= 2, 3! =6, 4! =24, 5! =120, 6! = 720 and digits larger than 6 can be excluded since 7! =5040 is already a four-digit number. We can also exclude 6, because if there is one 6 among the digits, then *A*> 6! =720 and the hundreds digit is larger than 7. The digit 5 has to be included, since *A*≤ 4! +4! +4!=72, which has only two digits. Because 5!+ 5!+ 5!=360, the first digit is at most 3. Because 3!+ 5!+ 5!=246, the first digit is at most 2.
Now 2!+ 5!+ 5!=242<255 and 2!+ 4!+ 5!=146 so the first digit is 1. Given the two of the digits are 1 and 5, we try to find the third digit *n*. By checking *n*=0, 1, 2, 3, 4 , *A*= 121 +*n*!, we find only *n*=4 works, that is 145= 1!+ 4!+ 5!. ***Answer:*** 145

5. All six faces of a cube are completely painted. It is cut into 64 identical cubes. One of these cubes is chosen at random and rolled. Find the probability that none of the five faces showing is painted.

【Solution】

There is a 1/64 probability of choosing any one of the small cubes. Since none of the five visible faces is painted, the chosen cube either has no painted faces or has one painted face, which is out of light (with probability1/6) . There are 2×2×2=8 small cubes with no painted faces (from the middle of the large cube), and 6×4=24 with one painted face (four from each of the six large faces). The probability of no painted face being visible is therefore $\frac{8}{64}\times 1+\frac{24}{64}\times\frac{1}{6}=\frac{3}{16}=18.75\%$. ***Answer:*** $\frac{3}{16}=18.75\%$.

6. Solve the equation $\sqrt{4+\sqrt{4-\sqrt{4+\sqrt{4-x}}}}=x$, where all square roots are taken to be positive.

【Solution】

Consider $f(x)=\sqrt{4+\sqrt{4-x}}$, then $f(f(x))=\sqrt{4+\sqrt{4-\sqrt{4+\sqrt{4-x}}}}$.

A solution to $f(x)=x$, if it exists, will also be a solution to $f(f(x))=x$.

Now we try to solving $f(x)=x$.

$f(x)=\sqrt{4+\sqrt{4-x}}=x$, let $y=\sqrt{4-x}$, then $y^2=4-x$.

We also have $x=\sqrt{4+y}$, from which $x^2=4+y$.

Subtracting, we have $x^2-y^2=x+y$. Hence $(x+y)(x-y-1)=0$.

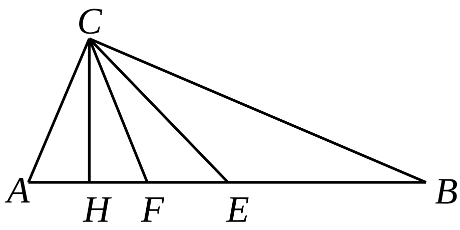
Since $x\ge 0$ and $y\ge 0$, $x+y=0$ implies *x*=*y*=0, which does not satisfy $f(x)=x$.

Therefore we take $x-y-1=0$, or $y=x-1$.

Substituting into $x^2=4+y$, we obtain $x^2=x+3$, or $x^2-x-3=0$.

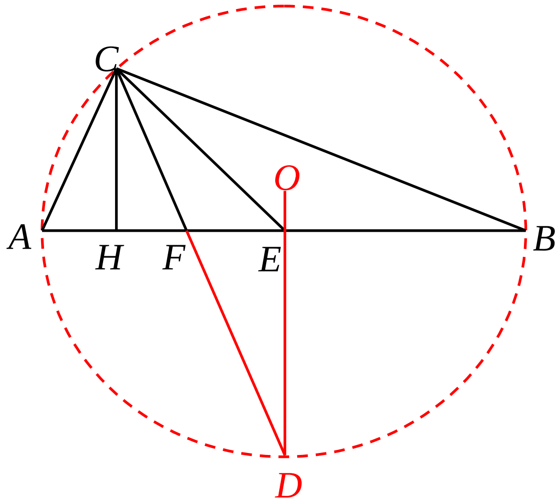
Rejecting the negative root, we have $x=\frac{1+\sqrt{13}}{2}$. ***Answer:*** $\frac{1+\sqrt{13}}{2}$

7. In the figure, *BC*>*AC*, *AE*=*EB*, *CH*⊥ *AB*, ∠*ACF*=∠*FCB* and ∠*HCF*=∠*FCE*. Find the measure of ∠*ACB*, in degrees.



【Solution 1】

Draw the perpendicular bisector of *AB* intersects the circumcircle of triangle *ABC* which lies on the other side of line *AB* than vertex *C* at point *D*, let *O* be the circumcentre. Since the arcs *AD* and *BD* are equal, Hence ∠*ACD*=∠*BCD*, ray *CD* is the bisector of ∠*ACB* and hence *F* lies on *CD*.



We have ∠*HCF*=∠*FCE*, in other words, ∠*HCD*=∠*DCE*.

Since *CH*//*DE*, ∠*EDC*=∠*HCD*=∠*DCE*, we get *CDE* is an isosceles triangle: *CE*=*DE*.

Note that triangle *CDO* also is isosceles: *CO*=*DO*. Since *O* lies on line *DE*, hence points *E* and point *O* are coincide.

The circumcentre coincides with the midpoint of side *AB* only if *AB* is the diameter of the circumcircle. Thus ∠*ACB*=90°

【Solution 2】

Let *a*, *b* and *c* be the lengths of sides *BC*, *CA* and *AB* and let α, β and γ be the sizes of angles *A*, *B* and *C*, respectively.

Since ∠*ACF*=∠*FCB* and ∠*HCF*=∠*FCE*, that is ∠*ACH*=∠*BCE* $=90°-\alpha$.

Hence ∠*ACE* =∠*ACB*− ∠*BCE* $=\gamma-(90°-\alpha)=\alpha+\gamma-90°=90°-\beta$.

Applying the Law of Sines to triangles *ACE* and *BCE*:

$$\frac{AE}{CE}=\frac{\sin\angle ACE}{\sin\angle CAE}=\frac{\sin(90°-\beta)}{\sin\alpha}=\frac{\cos\beta}{\sin\alpha}\qquad(1)$$

$$\frac{BE}{CE}=\frac{\sin\angle BCE}{\sin\angle CBE}=\frac{\sin(90°-\alpha)}{\sin\beta}=\frac{\cos\alpha}{\sin\beta}\qquad(2)$$

Since *E* is the midpoint of *AB*, we obtain $\frac{\cos\beta}{\sin\alpha}=\frac{\cos\alpha}{\sin\beta}$, $\sin\alpha\cos\alpha=\sin\beta\cos\beta$, hence $\sin 2\alpha=\sin 2\beta$.Since *BC*>*AC*, hence $\alpha\neq\beta$ and so $2\alpha+2\beta=180°$, which means $\gamma=90°$.

Answer: 90°

8. How many ordered triple (*x*, *y*, *z*) of integers satisfy *xyz* = 2012?

【Solution】

Since $2012=2^2\times503$, consider $|x|=2^{p_1}\times503^{q_1}$, $|y|=2^{p_2}\times503^{q_2}$ and $|z|=2^{p_3}\times503^{q_3}$, where p_i, q_j are non-negative integers such that $p_1+p_2+p_3=2$ and $q_1+q_2+q_3=1$. Hence the number of positive integer solutions to the equation *xyz*=2012 is $C_2^4\times C_2^3=18$.

Together with the 4 possible distribution of the signs: (+ , + , +), (+ , − , −), (− , + , −) and (− , − , +), there are 18 × 4 = 72 integer solutions.

ANS: 72