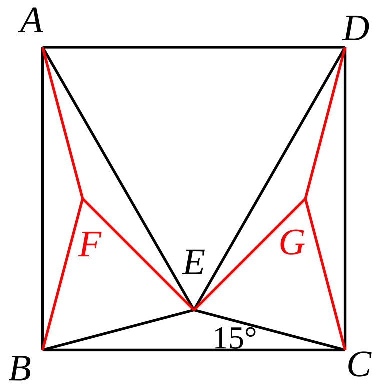
International Young Mathematicians' Convention (IYMC)
2012 Team Contest – Senior level

1. In the figure, E is a point inside the square $ABCD$ such that $BE=CE$ and $\angle BEC = 150°$. Prove that $\triangle ADE$ is an equilateral triangle.



【Solution 1】

Draw $\triangle FAB$ in left part of the square so that $\triangle FAB \cong \triangle EBC$. Thus $BF = BE$.

$\angle FBE = 90° - 15° - 15° = 60°$.

Since $BF = BE$, hence $\triangle BEF$ is an equilateral triangle.

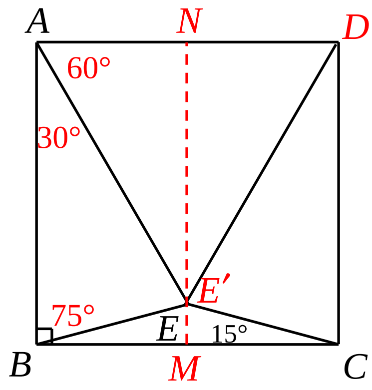
Thus $AF = FE$.

We get $\angle AFE = 360° - 60° - 150° = 150°$.

Hence $\angle FAE = \angle AEF = 15°$, and $\angle EAD = 90° - 15° - 15° = 60°$.

Since $AB=AE=AD$, hence we get ADE is an equilateral triangle.

【Solution 2】



Let M and N be the midpoint of BC and AD, respectively. Thus MN is the perpendicular bisector of BC and hence E is on MN.

We can find a point E' on MN such that $AE' = AB$.

Then $DE' = AE' = AB = AD$, i.e. $\triangle AE'D$ is an equilateral triangle.

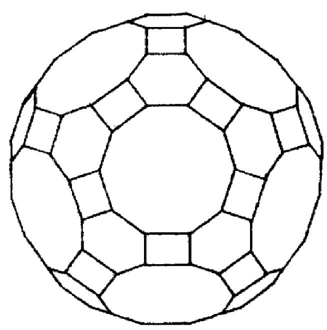
So $\angle BAE' = 90° - 60° = 30°$.

We get that $\angle ABE' = \angle AE'B = \dfrac{180° - 30°}{2} = 75°$.

Hence $\angle E'BC = 90° - \angle ABE' = 15°$.

Thus E and E' are coincide , we get $\triangle ADE$ is an equilateral triangle.

2. The figure shows a polyhedron in which each vertex lies on one regular decagon, one regular hexagon and one square. If this polyhedron have V vertices, E edges, F faces, find the value of $V+E+F$.



【Solution】

The sum of the angles that meet at each vertex is:

$$144 + 120 + 90 = 360 - \frac{720}{V}$$

$$V = 120$$

Suppose there are x decagons, y hexagons, and z squares.

Three faces intersect at each vertex, so we have $120 \times 3 = 10x + 6y + 4z$.

Each decagon is adjacent to five hexagons, and each hexagon is adjacent to three decagons. Therefore, we get $5x=3y$.

Each decagon is adjacent to five squares, and each square is adjacent to two decagons. Therefore, we get $5x=2z$.

Hence we know x=12, y=20 and z=30.

So V=120, E=(12×10+6×20+4×30)÷2=180 and F=12+20+30=62.

Thus V+E+F=120+180+62=362.

Answer: 362

3. Five boxes contain 120 coins in all. Some are golden coins, some are silver coins and the rest are copper coins. Any two boxes contain less than 30 golden coins. Any three boxes contain less than 20 silver coins. Prove that some four boxes contain at least 15 copper coins.

【Proof】

Ordering all boxes in increasing order with respect to the amount of golden coins they contain. The fourth and fifth boxes contain less than 30 golden coins so the fourth box contains not more than 14 golden coins.

Thus so does first, second and third boxes. Then total amount of golden coins is at most 71 = 14+14+14+29.

In a similar way order all boxes in increasing order with respect to the amount of silver coins they contain. The third, fourth and fifth boxes contain less than 20 silver coins so the third one contains not more than 6 silver coins.

Thus so does first and second boxes. Then total amount of silver coins is at most 31 = 6+6+19. We get the total amount of copper coins is at least 120－71－31=18.

Let's order all boxes in increasing order with respect to the amount of copper coins they contain. The first box contain no more than 18÷5=3.6 copper coins so the other four boxes contain at least 15 coins.

4. A man uses a gold chain consisted of 159 links to repay a debt. By agreement, he must hand in one link per week. He may hand in more links if he can get exact change. The debt is paid when all 159 links have been handed in. What is the minimum number of links that must be broken?

【Solution】

Note that when a link in the center of the chain is broken, three pieces are obtained: a one-link piece and two other pieces.

If he only broken 1 link, then there must have a 2-link piece and hence another one is a 156-link piece. Thus he can't get 4.

If he only broken 2 links, then there must have a 3-link piece, a 6-link piece and hence the other one is a 148-link piece. Thus he can't get 12.

If he only broken 3 links, then there must have a 4-link piece, a 8-link piece, a 16-link piece and hence the other one is a 128-link piece. Thus he can't get 32.

So the minimum number of links that must broken is 4.

If he broken the 6^{th}, 17^{th}, 38^{th} and 79^{th} link. This would result in 4 one-link pieces, one 5-link piece, one 10-link piece, one 20-link piece, one 40-link piece, and one 80-link piece.

Using those pieces he can pay any debt from 1 to 159.

Answer: 4 links

5. The sequence $\{ f(1), f(2), f(3), \ldots. \}$ of increasing positive integers satisfies $f(f(n)) = 3n$. Find the value of $f(2012)$.

【Solution 1】

We prove the following lemma.

<Lemma> For n=0, 1, 2, …, (1) $f(3^n) = 2\times 3^n$ and (2) $f(2\times 3^n) = 3^{n+1}$.

We use induction. For n=0, note that $f(1) \neq 1$, otherwise $3 = f(f(1)) = f(1) = 1$, which is impossible. Since $f: \mathbb{N} \to \mathbb{N}$, $f(1) > 1$, and $f(n+1) > f(n)$, f is increasing. Thus $1 < f(1) < f(f(1)) = 3$ or $f(1) = 2$. Hence $f(2) = f(f(1)) = 3$.

Suppose that for some positive integer $n \geq 1$, $f(3^n) = 2\times 3^n$ and $f(2\times 3^n) = 3^{n+1}$.

Then $f(3^{n+1}) = f(f(2\times 3^n)) = 2\times 3^{n+1}$ and $f(2\times 3^{n+1}) = f(f(3^{n+1})) = 3^{n+2}$ as desired.

This completers the induction.

There are $3^n - 1$ integers m such that $3^n < m < 2\times 3^n$ and there are $3^n - 1$ integers m' such that $f(3^n) = 2\times 3^n < m' < 3^{n+1} = f(2\times 3^n)$.

Since f is an increasing function, $f(3^n + m) = 2\times 3^n + m$ for $0 \leq m \leq 3^n$. Therefore

$$f(2\times 3^n + m) = f(f(3^n + m)) = 3(3^n + m) \text{ for } 0 \leq m \leq 3^n.$$

Hence $f(2012) = f(2\times 3^6 + 554) = 3(3^6 + 554) = 3849$

【Solution 2】

For integer n, let $n_{(3)} = a_1 a_2 \cdots a_l$ denote the base 3 representation of n. Using similar induction as in the first solution, we can prove that

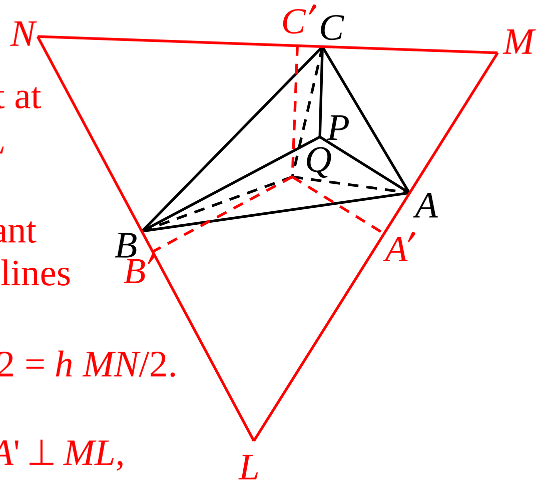
$$f(n)_{(3)} = \begin{cases} 2a_2\cdots a_l & \text{if } a_1 = 1 \\ 1a_2\cdots a_l 0 & \text{if } a_1 = 2 \end{cases}$$

Since $2012_{(3)} = 2202112$, $f(2012)_{(3)} = 12021120$ or

$f(2012) = 1\times 3^7 + 2\times 3^6 + 2\times 3^4 + 1\times 3^3 + 1\times 3^2 + 2\times 3^1 = 3849$. ANS: 3849

6. In the figure, P and Q are two distinct points inside the acute triangle ABC. If $\angle APB = \angle BPC = \angle CPA = 120°$. Prove that $PA+PB+PC<QA+QB+QC$.

【Solution】



Find perpendicular lines to PA, PB, PC through points A, B, C; and the perpendicular lines intersect at M, N, L. We know that the interior angles of ΔMNL are all 60°, hence ΔMNL is an equilateral triangle.

Since any point in an equilateral triangle has constant sum of distances from the point to 3 perpendicular lines (If ΔMNL has height h, since $a\Delta MNL$ = $a\Delta PLM + a\Delta PMN + a\Delta PNL$, $(PA + PB + PC)\, MN/2 = h\, MN/2$. So we know $PA + PB + PC = h$ is constant.)

The point Q is different to P, through Q we find $QA' \perp ML$, $QB' \perp NL$, and $QC' \perp MN$.

By properties of right triangles: $QA' < QA$, $QB' < QB$, and $QC' < QC$, then $PA + PB + PC = QA' + QB' + QC' < QA + QB + QC$.