International Young Mathematicians' Convention
Junior Level

Individual Contest

1. Suppose every term in the sequence 1, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 2, 1, … is either 1 or 2. If there are exactly $(2k-1)$ twos between the k-th one and the $(k+1)$-th one, find the sum of its first 2016 terms. **【Submitted by Philippines】**

Solution

Let's divide this sequence into a series of groups. Each group starts with 1 and is followed by as many 2s as possible: {1, 2}, {1, 2, 2, 2}, {1, 2, 2, 2, 2, 2}, {1, 2, 2, …, 2}. It is clear that the k-th group contains $2k$ terms whose sum equals $1+(2k-1)\times 2=4k-1$. The numbers of terms in these groups form an arithmetic sequence. The total number of terms in first n groups equal

$$2\times 1+2\times 2+...+2\times n=2\times\frac{n(n+1)}{2}=n\times(n+1)$$

Simplifying $n\times(n+1)\le 2016$, we get $n\le 44$. This means that 2016 terms in the original sequence cover 44 complete group and 1 partial group. Hence there are 45 ones in the first 2016 terms and $(2016-45)$ twos. Therefore, their sum equal $45\times 1+(2016-45)\times 2=3987$.

ANS: 3987

2. Find all ordered triples (x, y, z) of integers satisfying $x^2+y^2+z^2+3<xy+3y+2z$. **【Submitted by Philippines】**

Solution

First express the inequality as $(x-\frac{y}{2})^2+\frac{3}{4}(y-2)^2+(z-1)^2<1$.

Since x, y, and z are integers, this implies z = 1 and y = 1, 2, or 3.
When y = 1, we obtain $x^2-x<0$, which has no integer solutions.
When y = 2, we obtain $x^2-2x<0$, whose only integer solution is x = 1.
When y = 3, we obtain $x^2-3x+2<0$, which has no integer solutions.

ANS: (1, 2, 1)

3. There are 2016 bus in a row. Each weighs an integral numbers of kilograms. Except for the rightmost one, the sum of the weight of each bus and twice the weight of its right neighbour is 36000 kg. Determine the weight of the rightmost bus in kg.

Solution

Let the cars be numbered 1 to 2016 from left to right. Let the weight of the *i*-th bus be $12000 + x_i$ kg. Then for $1 \le i \le 2015$, $(12000 + x_i) + 2(12000 + x_{i+1}) = 36000$, so that $x_i = -2x_{i+1}$. Hence $x_1 = -2x_2 = 2^2 x_3 = \cdots = 2^{2014} x_{2015} = -2^{2015} x_{2016}$. Now $2^{2015} > 2^{16} = 65536 > 36000$. We cannot have $x_{2016} > 0$ as otherwise the first car will have negative weight. We cannot have $x_{2016} < 0$ either as otherwise the first car will have weight exceeding 36000 kilograms. Hence $x_{2016} = 0$, which implies that $x_i = 0$ for all *i*. Thus each bus weighs 12000 kg.

ANS: 12000 kg

4. Let *n* be a positive integer which is not less than 2016 so that $\dfrac{n-2016}{2116-n}$ is a positive square integral number. Find the sum of all of the possible values of *n*.

Solution

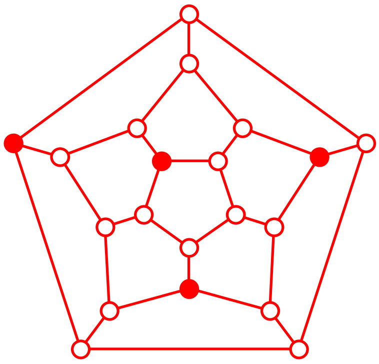
Suppose $2116 - n = a$, then $\dfrac{n-2016}{2116-n} = \dfrac{100-a}{a} = \dfrac{100}{a} - 1$. So *a* is a factor of 100. Since $\dfrac{n-2016}{2116-n}$ is a square, we can suppose $\dfrac{100}{a} - 1 = m^2$, where *m* is a positive integer. Thus $\dfrac{100}{a} = m^2 + 1$. Observe that only when *a* = 50, 20, 10 or 2, *m* =1, 2, 3 or 7, respectively. So there are 4 possible values of *n* so that we can make $\dfrac{n-2016}{2116-n}$ is a square number. They are 2066, 2096, 2106 and 2114. Thus the sum is 8382.

ANS: 8382

5. All vertices of a dodecahedron are white initially. Some vertices of the dodecahedron are to be painted red so that each face contains a red vertex. What is the largest number of vertices that are white?

Solution

Since there are 12 faces and each vertex lies on 3 of them, the minimum is 4. Because there are 20 vertices of a dodecahedron, the largest number of vertices that are not red is $20 - 4 = 16$. This can be accomplished as shown in the diagram below where the dodecahedron is represented by a planar graph and the chosen vertices are represented by red circles.



Answer: 16 vertices

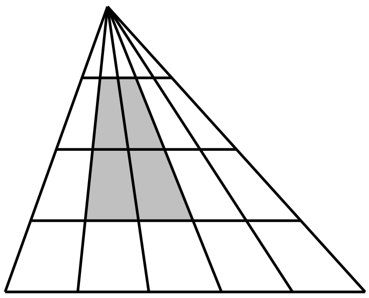
6. A 9-digit number consists of the digits 1, 2, 3, …, 9 in some order. Consider all triples of consecutive digits and find the sum of these seven 3-digit numbers. What is the smallest possible value of this sum?

Solution

Let the digits be *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h* and *i* in that order. Then *a* only appears as the leading digit of one three-digit number. Hence it contributes 100*a* to the sum. Similarly, the contribution of *b* is 110*b*, that of *h* is 11*h*, that of *i* is *i*, and that of any of the five digits in the middle is 111 times itself. It follows that we must have *a* = 7, *b* = 6, *h* = 8 and *i* = 9, while *c*, *d*, *e*, *f* and *g* can be any permutation of 1, 2, 3, 4 and 5. The smallest value of the sum is $700+660+111(1+2+3+4+5)+88+9=3122$.

Answer: 3122

7. The diagram below shows a triangle divided by seven lines. Four of them join the top vertex to points which divide the base into five equal parts. Three of them are parallel to the base and evenly spaced. If the area of the whole triangle is 900 cm^2, what is the area, in cm^2, of the shaded quadrilateral?



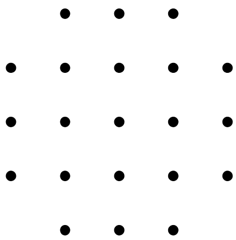
【Solution】

The four lines from the top vertex divide the triangle into five narrow triangles with same area. The three lines parallel to the base divide the triangle into four horizontal strips. The top strip, the top two strips, the top three strips and all four strips form four similar triangles whose areas are in the ratio $1^2 : 2^2 : 3^2 : 4^2$. Hence the areas of the four strips are in the ratio $1^2 : 2^2-1^2 : 3^2-2^2 : 4^2-3^2=1:3:5:7$. Hence the combined area of the middle two strips is equal to the combined area of the top and the bottom strips. It follows that the area of the shaded quadrilateral is equal to

$900\times\frac{2}{5}\times\frac{1}{2}=180\,\text{cm}^2$.

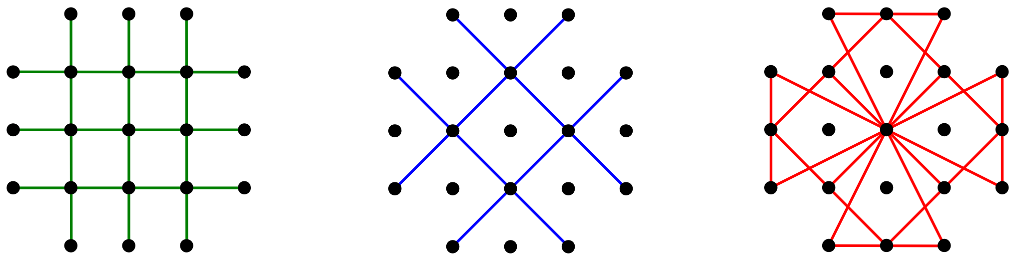
ANS: 180 cm^2

8. The four dots at the corners of a 5 by 5 array are removed, as shown in the diagram below. In how many different ways can we choose 3 of the remaining 21 points such that they lie on a straight line?



Solution

In the diagram below, there are 6 green lines each with 5 dots, 4 blue lines each with 4 dots and 14 red lines each with 3 dots. Hence the total number of ways is the sum of $C_3^5\times6=60$, $C_3^4\times4=16$ and $C_3^3\times14=14$. The total is $60+16+14=90$.



ANS: 90 ways