International Young Mathematicians' Convention

Junior Level

Team Contest

1. A function $f(x, y)$ satisfies $f(x, y)+f(x, z)=f(x, y+z+3)$ and $f(x, y)+f(z, y)=f(x+z+4, y)$. Find the value of $f(2016, 2017)$ if $f(1,1)=1$.

Solution

We have

$$\begin{aligned} f(2016, y) &= f(1, y)+f(2011, y) \\ &= 2\times f(1, y)+f(2006, y) \\ &\vdots \\ &= 403\times f(1, y)+f(1, y) \\ &= 404\times f(1, y) \end{aligned}$$

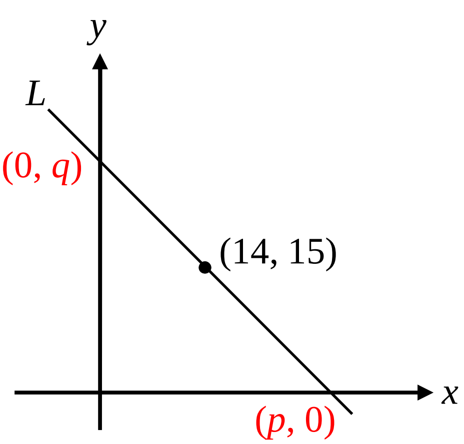
So $f(2016, 2017)=404\times f(1, 2017)$. And we also have

$$\begin{aligned} f(1, 2017) &= f(1,1)+f(1, 2013) \\ &= 2\times f(1,1)+f(1, 2009) \\ &\vdots \\ &= 504\times f(1,1)+f(1,1) \\ &= 505\times f(1,1) \end{aligned}$$

Hence $f(2016, 2017)=404\times 505\times f(1,1)=204020$.

ANS: 204020

2. In a rectangular coordinate plane, straight line L passes through the point (14, 15), the x-coordinate in the intersection point of L and x-axis is a prime number, the y-coordinate in the intersection point of L and y-axis is a non-zero positive integer. How many possible straight line L will meet all the given conditions? 【**Submitted by Philippines**】



Solution

Let the intersection point of L and x-axis denoted as $(p, 0)$, the intersection point of L and y-axis denoted as $(0, q)$, where p is a prime number greater than 14 and q is a nonzero positive integer greater than 15.

Since (p, 0), (14, 15), (0, q) must be collinear, then the slope of L is $\frac{0-15}{p-14}=\frac{15-q}{14-0}$.

So that $(p-14)(15-q)=(-15)\times 14=-210$, which is equivalent as $(p-14)(q-15)=210$. Observe that $p-14$ is an odd number since p is an odd prime number. It follows that

$p-14$	1	3	5	7	15	21	35	105
$q-15$	210	70	42	30	14	10	6	2

or

p	15	17	19	21	29	35	49	119
q	225	85	57	45	29	25	21	17

Hence, (p, q) = (17, 85), (19, 57) or (29, 29). There are 3 possible solutions.

ANS: 3

【Marking Scheme】

- Describe the slope of L, 10 marks.
- Find the relation $(p-14)(q-15)=210$, 10 marks.
- List all correct integral pairs (p, q), 10 marks.
  Note that if the student didn't observe that $p-14$ is an odd number, then he/she has to list 16 pairs.
- 3 marks for each possible solution.
- Get all of the possible solutions without wrong solutions, 10 marks.

3. A, B, C, D and E participated in a marathon. At the start, A is leading, B is in the second place, C is in the middle, D is in the fourth place and E is last. During the marathon, A trades place with the other four runners a total of 11 times, while B does that 11 times, C does that 22 times and E does that 44 times. If A finished ahead of D, who is in the second place among these five runners at the finish?

Solution:

When a runner trade places in a race, an odd numbered position becomes even numbered, and vice versa.

Since A starts in an odd-numbered place and trades places an odd number of times, he must finish in an even-numbered position.

Since B starts in an even-numbered place and trades places an odd number of times, he must finish in an odd-numbered position.

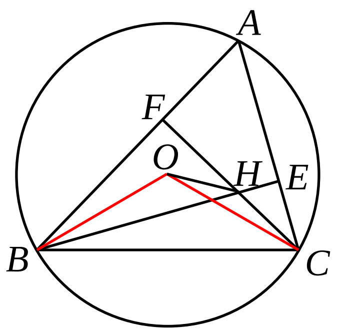
Since C starts in an odd-numbered place and trades places an even number of times, he must finish in an odd-numbered position.

Since E starts in an odd-numbered place and trades places an even number of times, he must finish in an odd-numbered position.

So D must finish in an even-numbered position. Since A finished ahead of D, A is in the second place and D is in the fourth place at the finish.

Answer: A

4. In the acute triangle ABC, $\angle A = 60°$. Two altitudes, BE and CF, intersect at point H. Point O is the circucentre of triangle ABC, as shown in the diagram below. Find the measure of $\angle FHO$, in degrees.



Solution

Connect OB and OC. Observe that

$\angle BOC = 2\angle BAC = 120°$.

Also $\angle BHC = \angle FHE = 180° - \angle BAC = 180° - 60° = 120°$.

Hence $BCHO$ is cyclic and $\angle OHB = \angle OCB = 30°$ since $OB = OC$.

On the other hand, $\angle BHF = 180° - \angle BHC = 180° - 120° = 60°$. It follows that

$\angle FHO = 60° - 30° = 30°$.

Answer: $30°$

【Marking Scheme】

- Observe that $\angle BOC = 120°$, 10 marks.
- Observe that $\angle BHC = 120°$, 10 marks.
- Observe that $\angle OHB = \angle OCB = 30°$, 10 marks.
- Conclude that $\angle BHF = 60°$, 5 marks.
- Conclude that $\angle FHO = 30°$, 5 marks.

5. A code of the coffer is an arrangement of numbers satisfied that the differences between neighbours are all different. For example, the numbers

$$\boxed{1}\ \boxed{4}\ \boxed{2}\ \boxed{3}$$

have differences 3, 2 and 1 — all different. Now the code is the numbers from 1 to 6 that are arranged with the condition, and with 4 in the fourth position from left to right,

$$\boxed{}\ \boxed{}\ \boxed{}\ \boxed{4}\ \boxed{}\ \boxed{}$$

Find the sum of the possible values of the rightmost position of the code.

Solution

When placing the numbers 1, 2, 3, 4, 5 and 6 in six boxes, there are five differences, which we will write between the boxes:

$$\boxed{?}\,^{?}\,\boxed{?}\,^{?}\,\boxed{?}\,^{?}\,\boxed{4}\,^{?}\,\boxed{?}\,^{?}\,\boxed{?}$$

Since the only possible differences are 1, 2, 3, 4 and 5, each must appear once. The difference of 5 must come from the pair

$$\boxed{1}\,^{5}\,\boxed{6} \text{ (possibly reversed)}$$

and the difference of 4 from either

$$\boxed{1}\,^{4}\,\boxed{5} \text{ or } \boxed{2}\,^{4}\,\boxed{6}$$

These always have one number in common, so they must be in the first three boxes. That is, one of

$\boxed{1}^{5}\boxed{6}^{4}\boxed{2}$ $\boxed{2}^{4}\boxed{6}^{5}\boxed{1}$ $\boxed{6}^{5}\boxed{1}^{4}\boxed{5}$ $\boxed{5}^{4}\boxed{1}^{5}\boxed{6}$

The difference of 3 must come from one of

$$\boxed{1}^{3}\boxed{4} 、 \boxed{2}^{3}\boxed{5} \text{ or } \boxed{3}^{3}\boxed{6}$$

Observe that the last two cases can’t be in the last two boxes since 6 and one of 2 and 5 must be in the first three boxes. So, the only way to fit one of these with one of the four possibilities for the first three boxes is

$$\boxed{2}^{4}\boxed{6}^{5}\boxed{1}^{3}\boxed{4}^{?}\boxed{?}^{?}\boxed{?}$$

The final two numbers can be placed either way around, giving two codes:

$$\boxed{2}^{4}\boxed{6}^{5}\boxed{1}^{3}\boxed{4}^{2}\boxed{3}^{1}\boxed{5} \text{ and } \boxed{2}^{4}\boxed{6}^{5}\boxed{1}^{3}\boxed{4}^{1}\boxed{5}^{2}\boxed{3}$$

So the sum of the possible values of the rightmost position of the code is $5+3=8$.

Answer: 8

6. A positive integer consists of distinct digits each of which is a divisor of the number. What is the largest number that fit the condition?

Solution

Clearly, none of the digits may be 0. If one of them is 5, then the units digit of the number must be 5, and the number will not be divisible by 2, 4, 6 or 8. This leaves a maximum of 5 different digits, which we shall see is short of the maximum. Thus it is better to leave out 5. Note that $1 + 2 + 3 + 4 + 6 + 7 + 8 + 9 = 40$, which is not a multiple of 3. The first multiple of 9 below 40 is 36. By leaving out 4, we may have as many as 7 different digits.

Now we want to find the largest 7-digit number so we can assume the last three digits are formed by 1, 2 and 3 first. Since the 7-digit number is a multiple of 8, 312 is the end of the 7-digit. Since 9876312 is not a multiple of 7 and $9867312 = 7 \times 1409616$, the largest number that fit the condition is 9867312.

Answer: 9867312

【Marking Scheme】

- Observe that none of the digits may be 0, 5 marks.
- Observe that it is better to leave out 5, 5 marks.
- Observe that it is better to leave out 4, 10 marks.
- Observe that 312 is the end of the 7-digit, 10 marks.
- Conclude that the answer is 9867312, 10 marks.