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International Young Mathematicians' Convention
Senior Level
Individual Contest

1. Find the number of all real solutions of the system of equations

$$(x_3+x_4+x_5)^5=3888x_1$$
$$(x_4+x_5+x_1)^5=3888x_2$$
$$(x_5+x_1+x_2)^5=3888x_3$$
$$(x_1+x_2+x_3)^5=3888x_4$$
$$(x_2+x_3+x_4)^5=3888x_5$$

Solution

By symmetry, we may assume that $x_1=\max\{x_1,\ x_2,\ x_3,\ x_4,\ x_5\}$. Then

$$(x_3+x_4+x_5)^5=3888x_1\ge 3888x_2=(x_4+x_5+x_1)^5$$

This implies that $x_3+x_4+x_5\ge x_4+x_5+x_1$ so that $x_3\ge x_1$. Hence $x_3=x_1$. Similarly, we can prove that $x_4=x_2=x_5=x_3=x_1$. The real roots of $(3x_1)^5=3888x_1$ are $x_1=0$, 2 and -2. Hence the system has three solutions, $(x_1,\ x_2,\ x_3,\ x_4,\ x_5)$ = (0, 0, 0, 0, 0), (2, 2, 2, 2, 2) and $(-2,-2,-2,-2,-2)$.

Answer: 3

2. What is the simplified value of $\sqrt{5+\sqrt{5^2+\sqrt{5^4+\sqrt{5^8+\cdots}}}}$?【**Submitted by Philippines**】

Solution

Suppose $x=\sqrt{5+\sqrt{5^2+\sqrt{5^4+\sqrt{5^8+\cdots}}}}$, then

$$x^2=5+\sqrt{5^2+\sqrt{5^4+\sqrt{5^8+\cdots}}}$$
$$=5+\sqrt{5}\times\sqrt{5+\sqrt{5^2+\sqrt{5^4+\sqrt{5^8+\cdots}}}}$$
$$=5+\sqrt{5}x$$

i.e. $x^2-\sqrt{5}x-5=0$.

Since $\sqrt{5+\sqrt{5^2+\sqrt{5^4+\sqrt{5^8+\cdots}}}}>0$, $x=\dfrac{\sqrt{5}+\sqrt{5+20}}{2}=\dfrac{\sqrt{5}+5}{2}$.

ANS: $\dfrac{\sqrt{5}+5}{2}$

3. If a is a positive integer so that a^2+2016^2 is divisible by 2016a, find the number of the possible values of a.

Solution

Let p be any prime divisor of a. Then p divides 2016a, which in turn divides a^2+2016^2. Hence p is also a prime divisor of 2016. Suppose p^m is the highest power of p which divides a, and p^n is the highest power of p which divides 2016. We may assume that $m \ge n$. Now 2016a is divisible by p^{m+n} but a^2+2016^2 is not divisible by p^{2n+1}. Hence $m+n<2n+1$ or $m \le n$, so that $m=n$. Since p is an arbitrary prime divisor of a and 2016, we have $a=2016$. Hence there is only one possible value of a.

Answer: 1

4. Let $f(x)=\dfrac{x+20}{x}$ and $f_n(x)=f(f(\cdots(f(x))\cdots))$ be the n-fold composite off.

For example, $f_2(x)=\dfrac{\frac{x+20}{x}+20}{\frac{x+20}{x}}=\dfrac{21x+20}{x+20}$ and $f_3(x)=\dfrac{\frac{21x+20}{x+20}+20}{\frac{21x+20}{x+20}}=\dfrac{41x+420}{21x+20}$.

Let S be the complete set of real solutions of $f_n(x)=x$. What is the maximal number of the elements in S ?

Solution

We have, for all positive integral n, $f_n(x)=f(f_{n-1}(x))=\dfrac{f_{n-1}(x)+20}{f_{n-1}(x)}=x$.

Then $x\,f_{n-1}(x)=f_{n-1}(x)+20$, i.e. $f_{n-1}(x)=\dfrac{20}{x-1}=x$.

So $x^2-x-20=0$, i.e. $(x+4)(x-5)=0$. Solving the equation and we can get $x=-4$ or 5. Hence there are two elements in S and the maximal number is 5.

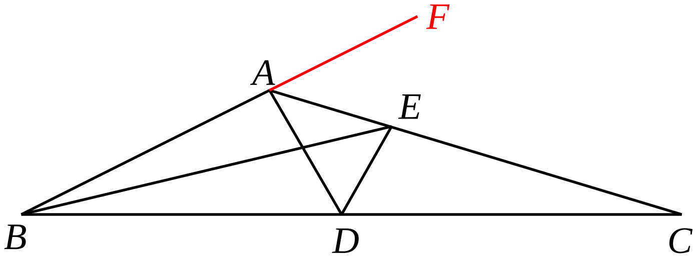
Answer: 5

5. D and E are points on the sides BC and CA, respectively, of triangle ABC. If $\angle ADC=130°$, $\angle BEA=25°$ and BE bisects $\angle ABC$, as shown in the diagram below. Find the measure of $\angle EDC$, in degrees.

Solution

Extend BA to F. We have

$$\begin{aligned}&\angle FAE-\angle DAE\\ &=(\angle ABC+\angle BCA)-(50°-\angle BCA)\\ &=2(\angle EBC+\angle ECB-25°)\\ &=0\end{aligned}$$



Hence E is an excentre of triangle BAD, so that $\angle EDC=\dfrac{1}{2}\angle ADC=65°$.

Answer: 65°

6. The sum of ten numbers on a circle is 2016. The sum of any three numbers in a row is at least 585. Determine the minimal number n such that for any such set of ten, none of them is greater than n.

Solution

Consider the largest number in any such set of ten numbers. The other nine form three triples in a row, each with sum at least 585. Hence the largest number is at most $2016-3\times585=261$. If 261, 261, 261, 63, 261, 261, 63, 261, 261 and 63 are arranged in cyclic order on a circle, the sum of any three numbers in a row is indeed at least 585, and the largest of them is 261. Hence the minimum value of n is 261.

Answer: 261

7. Anna tosses 2016 coins and Boris tosses 2017 coins. Whoever has more heads wins. If they have the same number of heads, then Anna wins. What is the probability of Anna winning?

Solution:

Let Boris first toss only 2016 coins. There are three possible outcomes.

(1) Anna has more heads than Boris.

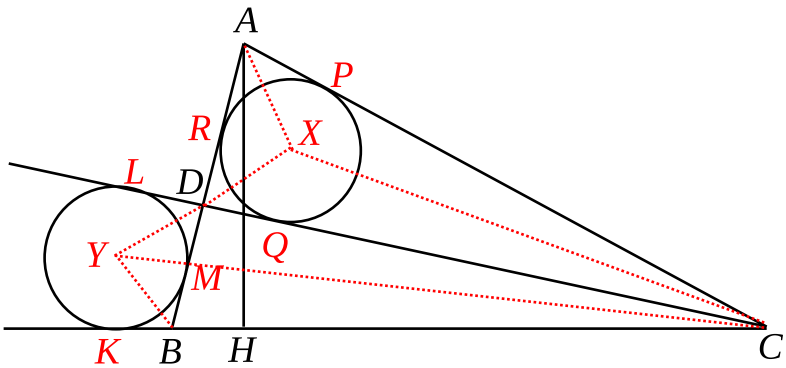
(2) Boris has more heads than Anna.

(3) They have the same number of heads.

If (1) occurs, then Anna wins, and if (2) occurs, then Boris wins, regardless of the outcome of Boris' last toss. If (3) occurs, then the winner will be decided by the outcome of Boris' last toss. If it is heads then Boris wins. If not, Anna wins. By symmetry, (1) and (2) are equally likely to occur and either player is equally likely to win if (3) occurs. Hence overall, Anna and Boris are equally likely to win, i.e. the probability of Anna winning is $\frac{1}{2}$.

Answer: $\frac{1}{2}$

8. In triangle ABC, $AC = BC$. D is a point on AB such that the inradius of triangle CAD is equal to the exradius of triangle BCD opposite C, as shown in the diagram below. If the length of the altitude AH is 36 cm, find the length of this common radius.



Solution

Let P, Q and R be the respective points of tangency of the incircle of ACD with AC, CD and DA. Let K, L and M be the respective points of tangency of the excircle of BCD with BC, CD and DB. Let X be the centre of the incircle of ACD and Y be the centre of the excircle of BCD. Let r be the common radius of the two circles. Then

$[ABC]=\frac{1}{2}AH\times BC$, $[ACD]=[AXC]+[CXD]+[DXA]=\frac{1}{2}r(AC+CD+DA)$ and $[BCD]=[BCY]+[CDY]-[DBY]=\frac{1}{2}r(BC+CD-DB)$. We have

$$\begin{aligned}
&AC+CD+DA+BC+CD-DB\\
&=2BC+CD+(DQ+QC)+(AR+RA)-(DM+MB)\\
&=2BC+(CD+DL)+(CP+PA)+(DM-DM)-BK\\
&=2BC+CL+AC-BK\\
&=3BC+CK-BK\\
&=4BC
\end{aligned}$$

It follows that $r=\frac{1}{4}AH=9\,\text{cm}$.

Answer: 9 cm