International Young Mathematicians' Convention

Senior Level

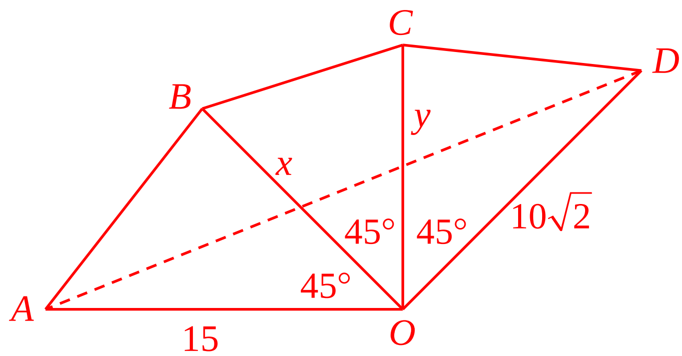
Team Contest

1. If x and y are positive real numbers, find the smallest value of

$$\sqrt{225-15\sqrt{2}x+x^2}+\sqrt{200-20y+y^2}+\sqrt{x^2-\sqrt{2}xy+y^2}.$$

Solution

Consider a pentagon $OABCD$ such that $OA=15$, $OB=x$, $OC=y$, $OD=10\sqrt{2}$ and $\angle AOB=\angle BOC=\angle COD=45°$.



Then by the cosine theorem,

$$AB=\sqrt{OA^2+OB^2-OA\times OB\times\cos 45°}$$
$$=\sqrt{225+x^2-15\sqrt{2}x},$$
$$BC=\sqrt{OB^2+OC^2-OB\times OC\times\cos 45°}$$
$$=\sqrt{x^2+y^2-\sqrt{2}xy}$$

and $CD=\sqrt{OC^2+OD^2-OC\times OD\times\cos 45°}=\sqrt{y^2+200-20y}$.

By the triangle inequality, the smallest value that $AB+BC+CD$ can have is AD, and by the cosine theorem it equals

$$AD=\sqrt{OA^2+OD^2-OA\times OD\times\cos 135°}=\sqrt{225+200+300}=\sqrt{725}=5\sqrt{29}.$$

Answer: $5\sqrt{29}$

2. The sum of 2016 real numbers is 2017 and each of them is less than $\frac{2017}{2015}$.

Prove that the sum of any two of the numbers is greater than or equal to $\frac{2017}{2015}$.

Proof

Suppose the sum of two of the numbers is less than $\frac{2017}{2015}$. Remove them to leave behind 2014 numbers with sum at least $2017-\frac{2017}{2015}=2015\times\frac{2017}{2015}-\frac{2017}{2015}=2014\times\frac{2017}{2015}$. By the Pigeonhole Principle, at least one of them must be greater than or equal to $\frac{2017}{2015}$. We have a contradiction.

【Marking Scheme】

- Using Counter–evidence method, 10 marks.
- Remove them to leave behind 2014 numbers with sum at least $2014\times\frac{2017}{2015}$, 10 marks.
- Conclude that at least one of them must be greater than or equal to $\frac{2017}{2015}$, 20 marks.

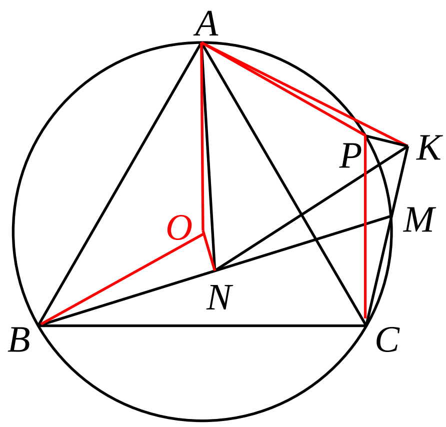
3. There are 2016 unit cubes, each of which can be painted black or white. How many values of *n* is it possible to construct an $n\times n\times n$ cube with n^3 unit cubes such that each cube shares a common face with exactly three cubes of the opposite colour?

Solution

The task is possible if and only if *n* is even. For odd *n*, construct a graph with n^3 vertices representing the unit cubes. Two vertices are joined by an edge if and only if the unit cubes they represent have different colours and share a common face. Since the number of vertices is odd, it is not possible for every vertex to have degree 3, as the total degree is even. For $n=2$, a checkerboard colouring works. For larger even *n*, the cube can be assembled from $2\times2\times2$ cubes in such a way that only cubes of the same colour come into contact in the merger. The possible values of *n* is 2, 4, 6, 8, 10 and 12 since $12^3=1728<2016<14^3=2744$. Hence there are 6 such values.

Answer: 6

4. *P* is the midpoint of the arc *AC* of the circumcircle of an equilateral triangle *ABC*. *M* is another point on this arc and *N* is the midpoint of *BM*. *K* is the projection of *P* on the line *MC*, as shown in the diagram below. If the length of *NA* is 19 cm, find the length of *NK* in cm.



Solution

Let *O* be the circumcentre of *ABC*. Connect *OA*, *ON*, *OB*, *PA*, *PC* and *KA*. Since $\angle ONB=90°=\angle PKC$, $\angle OBN=\angle PCK$ and *OB* = *PC*, triangles *BON* and *CPK* are congruent. It follows that $\angle BON=\angle CPK$ and *ON* = *PK*. Now $\angle AON=360°-\angle AOB-\angle BON=360°-\angle APC-\angle CPK=\angle APK$, so that triangles *AON* and *APK* are congruent. It follows that we have *AN* = *AK*. Now $\angle NAK=\angle OAK-\angle OAN=\angle OAK-\angle PAK=\angle OAP=60°$. Hence *ANK* is an equilateral triangle and $NK=NA=19$ cm

Answer: 19 cm

【Marking Scheme】

- Observe that triangles *BON* and *CPK* are congruent, 10 marks.

Or conclude that $\angle BON=\angle CPK$ and *ON* = *PK*, each 5 marks.

- Observe that triangles *AON* and *APK* are congruent, 10 marks.

Or conclude that *AN* = *AK*, 10 marks.

- Observe that *ANK* is an equilateral triangle, 10 marks
- Conclude that $NK = NA = 19$ cm, 10 marks.
- Correct answer without reasons, 0 mark.

5. Let S_n denote the *n*-th sequence so that every word in a sequence consists only of the letters *A* and *B*. The first word has only one letter *A*. For $k \ge 2$, the *k*-th word is obtained from the $(k-1)$-th by simultaneously replacing every *A* by *AAB* and every *B* by *A*. Then every word is an initial part of the next word. For example, $S_1 = A$, $S_2 = AAB$, $S_3 = AABAABA$ and $S_4 = AABAABAAABAABAAAB$. Find the number of *A*s in S_{10}.

Solution

Now S_n is obtained from $S_{n\text{-}1}$ by the replacement. We symbolize this as $t(S_{n-1}) = S_n$. Note that S_3 consists of two copies of S_2 and one copy of S_1 strung together in that order. We symbolize this as $S_3 = S_2 \circ S_2 \circ S_1$. We claim that for $n \ge 3$, $S_n = S_{n-1} \circ S_{n-1} \circ S_{n-2}$. This holds for *n* = 3. Suppose it holds for some $n \ge 3$. Then we have

$$\begin{aligned} S_{n+1} &= t(S_n) \\ &= t(S_{n-1} \circ S_{n-1} \circ S_{n-2}) \\ &= t(S_{n-1}) \circ t(S_{n-1}) \circ t(S_{n-2}) \\ &= S_n \circ S_n \circ S_{n-1} \end{aligned}$$

Hence our claim holds for all $n \ge 3$. Let a_n and b_n be the respective numbers of As and Bs in S_n. Then $a_1 = 1$, $b_1 = 0$ and, for $n \ge 2$, we have $a_n = 2a_{n-1} + b_{n-1}$ and $b_n = a_{n-1}$. These recurrence relations yields the following table.

n	1	2	3	4	5	6	7	8	9	10
a_n	1	2	5	12	29	70	169	408	985	2378
b_n	0	1	2	5	12	29	70	169	408	985
$a_n + b_n$	1	3	7	17	41	99	239	577	1393	3363

Answer: 2378

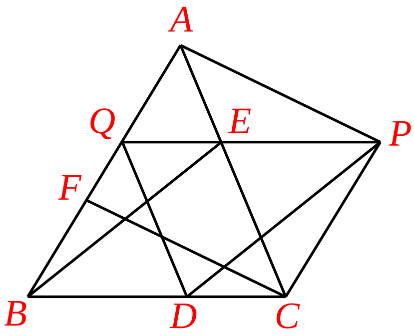
6. *D*, *E* and *F* are points on the sides *BC*, *CA* and *AB*, respectively, of triangle *ABC* such that *AD*, *BE* and *CF* are concurrent. The area of triangle *ABC* is 2016 cm^2. If there exists a point *P* such that both *BDPE* and *AFCP* are parallelograms, as shown in the diagram below. Find the area of triangle of *DEF*, in cm^2.

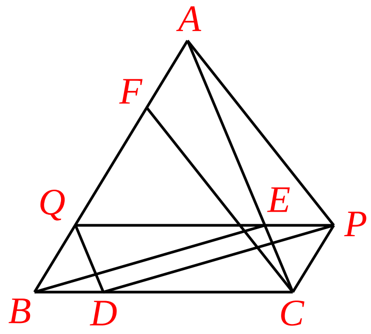
Solution

Extend *PE* to cut *AB* at *Q*. Then *BCPQ* and *DCEQ* are also parallelograms. Also, triangles *CEP* and *AEQ* are similar. Suppose $AE \ne CE$. We consider two cases.

Case 1. $AE < CE$.

Then $BD = PE > QE = CD$ and $AF = CP > AQ$. Hence Q lies on AF, as shown in the diagram below on the left, so that $AF = CP = BQ > BF$. Hence $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} > 1$, which contradicts Ceva's Theorem.





Case 2. $AE > CE$.

Then Q lies on BF, as shown in the diagram above on the right. We can prove as in Case 1 that $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} < 1$. This also contradicts Ceva's Theorem. It follows that $AE = CE$, so that triangles CEP and AEQ are congruent. Hence Q coincides with F, so that $AF = CP = BQ = BF$ and $BD = PE = QE = BD$.

Hence D, E and F are the respective midpoints of BC, CA and AB. Thus the area of triangle of DEF is $\frac{1}{4} \times 2016 = 504\,\text{cm}^2$.

Answer: 504 cm^2

【Marking Scheme】

- Observe that there are 2 cases, 5 marks.
- Complete the proof for $AE < CE$, 10 marks.
- Complete the proof for $AE > CE$, 10 marks.
- Conclude that D, E and F are the respective midpoints of BC, CA and AB, 10 marks.
- Conclude the correct answer, 5 marks.
- Just say D, E and F are the respective midpoints of BC, CA and AB and hence get the correct answer without reasons, 10 marks.