International Young Mathematicians' Convention

Junior level

Individual Contest

1. A farmer sold each of his peaches at a different price, the last one was sold for \$2.30. He computed that the average price of peaches was \$2.45. However, a customer returned a rotten peach and agreed buying it at a reduced price of \$1.58. The farmer re-computed the average price, which became \$2.42. What is the minimum number of peaches the farmer could have sold? 【**Submitted by *Jury***】

【Solution】

It does not matter which peach was sold at the reduced price, a reduction of \$0.72 brought the average down by \$0.03. Hence the number of peaches must be 24.

Answer: 24 peaches

2. Let S be a set of 6 integers taken from {1, 2, 3, …, 12} such that if a and b are both elements of S where $a < b$, then b is not a multiple of a. Find the total number of distinct S that satisfies the said conditions. 【**Submitted by *Thailand***】

【Solution】

If the smallest element of S is 1, we can include nothing else, so that won't work.
If the smallest element of S is 2, we would have to include every odd number except 1 to fill out the set, there are only2, 3, 5, 7, 11 five elements, so that won't work.
If the smallest element of S is 3, we can include 7, 11, and either 5 or 10 (which are always safe). There are only 3, 4, 5, 7, 11 five elements, so that won't work.
If the smallest element of S is 4, we can choose {4, 5, 6, 7, 9, 11}, {4, 6, 7, 9, 10, 11}.
If the smallest element of S is 5, we can choose {5, 6, 7, 8, 9, 11},{5, 7, 8, 9, 11,12}.
If the smallest element of S is 6, we can choose {6, 7, 8, 9, 10, 11}.
If the smallest element of S is 7, we can choose {7, 8, 9, 10, 11, 12}.
The smallest element of S cannot larger than 7. Hence we only have 6 possible S.

Answer: 6 sets

3. In a chess tournament, each player will play with every other player exactly once. A win is worth two points, a draw is worth one point, and a loss is worth zero. Two gifted students from an elementary school took part in a chess tournament at a nearby university and the combined score of both elementary school students is 13. If the scores of each university student are all the same, then what is the score that each university student got in the tournament?【**Submitted by *Jury***】

【Solution】

Let n university students participate in the tournament. Let the score for each be m, where m is a positive integer. The total number of games is $C_2^{n+2}=\dfrac{(n+2)(n+1)}{2}$ while the total score is $mn+13$. Hence $(n+2)(n+1)=mn+13$, so $n(n+3-m)=11$, which implies $n=1$ or 11. However, $n=1$ must be rejected as otherwise m would be negative. When $n=11$, we have $m=13$. Hence each university students got 13 points in the tournament.

Answer: 13 points

4. How many ordered pairs (m, n) of positive integers are there which satisfy the equation $m^2-2m=n^2+4n+2018$? 【**Submitted by *South Africa***】

【Solution】

$$m^2-2m=n^2+4n+2018$$
$$m^2-2m+1=n^2+4n+4+2015$$
$$(m-1)^2-(n+4)^2=2015=5\times13\times31$$
$$(m-n-3)(m+n+1)=2015=1\times2015=5\times403=13\times155=31\times65$$

If $m-n-3>m+n+3$, then $n<1$, it is impossible.

(i) $m-n-3=1$ and $m+n+1=2015$. So $m=1009$ and $n=1005$.

(ii) $m-n-3=5$ and $m+n+1=403$. So $m=205$ and $n=197$.

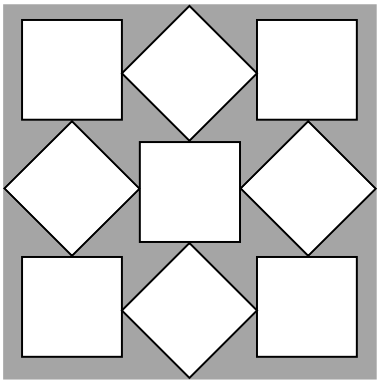
(iii) $m-n-3=13$ and $m+n+1=155$. So $m=85$ and $n=69$.

(iv) $m-n-3=31$ and $m+n+1=65$. So $m=49$ and $n=15$.

Hence the solutions are (1009, 1005), (205, 197), (85, 69) and (49, 15).

Answer: 4 ordered pairs

5. There are nine small squares, each with an area of 3 cm^2, are enclosed inside a larger square as shown in the diagram. If all the squares that are touching each other are coinciding with the midpoint of the side of the other square, then find the area, in cm^2, of the shaded region. 【**Submitted by *Philippines***】



【Solution】

Each of the small squares has side length $\sqrt{3}$ and diagonal length $\sqrt{3}\times\sqrt{2}=\sqrt{6}$. Since the large square has sides equal to the sum of the lengths of two of the diagonals and one of the sides of the small squares, the area of the large square is

$$(\sqrt{3}+2\sqrt{6})^2=3+4\sqrt{3}\times\sqrt{6}+4\times6=27+12\sqrt{2}.$$

Each small square has area 3, and there are nine small squares with a total area of 27. Thus, the desired area is $12\sqrt{2}=\sqrt{288}$ cm^2.

Answer: $12\sqrt{2}=\sqrt{288}$ cm^2

6. How many ways can a student schedule 3 mathematics subjects – Algebra, Geometry and Number Theory – in a seven-period school day if any one of these subjects is not allowed to be taken in consecutive periods? 【**Submitted by *Thailand***】

【Solution 1】

We can choose three periods for mathematics subjects such that no two mathematics subjects in consecutive periods:

Periods 1, 3, 5. Periods 1, 3, 6. Periods 1, 3, 7. Periods 1, 4, 6. Periods 1, 4, 7.
Periods 1, 5, 7. Periods 2, 4, 6. Periods 2, 4, 7. Periods 2, 5, 7. Periods 3, 5, 7.

There are 10 ways to place 3 non-distinguish classes into 7 periods such that no two classes are in consecutive periods. For each of these ways, there are $3!=6$ orderings of the classes among themselves. Therefore, there are $10\times6=60$ ways.

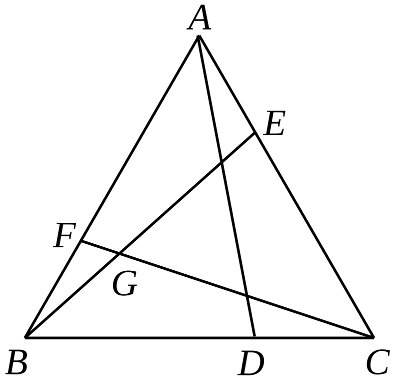
【Solution 2】

Denote mathematics subjects as M, and represent the space as O. Since no two mathematics subjects in consecutive periods, the arrangement must be in the format MOMOM.

We should put two more O's into it. The first O have 4 ways to put in, the second have 5 ways to put in, the O's are the same, whence we have $4\times5\div2=10$ ways. We have $3!=6$ orderings of the 3 distinct mathematics subjects, therefore, there are $10\times6=60$ ways.

Answer: 60 ways

7. In the figure, *ABC* is an equilateral triangle where *E* is a point on *CA*, *F* is a point on *AB* and *G* is a point of intersection of *BE* and *CF*. If triangle *BCG* has the same area as quadrilateral *AEGF*, then determine the measure, in degrees, of $\angle EGF$. 【**Submitted by *Jury***】



【Solution】

Since triangle *BCG* has the same area as the quadrilateral *AEGF*, triangle *BCF* has the same area as triangle *ABE* so that $BF = AE$. Let *D* be the point on *BC* such that $CD = BF$, and join *AD*. By symmetry, an equilateral triangle is formed by *AD*, *BE* and *CF*. It follows that $\angle EGF = 180° - 60° = 120°$.

Answer: 120°

8. There are six tokens having different weights namely 1g, 2g, 4g, 8g, 16g, and 32g. How many different ways can we get a weight of 21g using a regular two-sided weighing scale? (Note: Each token may be placed on either pan of the balance, and it is not necessary to use all the tokens in each weigh). **【Submitted by *Jury*】**

【Solution】

If the token of weight 32 g is not used, then the number of ways is clearly 5:

$$21=16+4+1=16+8-2-1=16+4+2-1=16+8-4+1=16+8-4+2-1.$$

If 32 g is use, it must be placed on the opposite pan. The number of ways is then same as the number of ways of balancing an object of weight $32-21=11$ g by using a set of tokens having weights 1 g, 2 g, 4 g, 8 g and 16 g. Observe that

$$11=8+2+1=8+4-1=16-8+2+1=16-8+4-1$$

$$=16-4-1=16-4-2+1=8+4-2+1=16-8+4-2+1$$

The number of ways is then 8. Hence balancing an object of weight 21 g is $5+8=13$ ways.

Answer: 13 ways