International Young Mathematicians' Convention

Junior level

Team Contest

1. A number is placed on each unit square of a checkerboard such that the numbers on any two unit squares that share a common side differ by 1. It is known that one of the unit squares is already filled up with the number 3, and another one with the number 17, find the total sum of the numbers filled on both diagonals of the checkerboard. 【**Submitted by *Jury***】

【Solution】

It takes 14 moves to go from one corner square to the opposite one, and since $17-3=14$, we must place 3 at one corner and 17 at the opposite corner. The numbers on the other squares are then uniquely determined, as shown in the figure below. The sum of the 16 numbers on the diagonals is 160.

10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10

Answer: 160

2. Let f be a function for all x and y, where x and y are integers. If $f(f(x)+y)-f(y+7)=x$ and $f(2)=5$, then what is the value of $f(2018)$? 【**Submitted by *South Africa***】

【Solution】

$$f(f(x)+y)-f(y+7)=x$$

When $x=2$,

$$f(f(2)+y)-f(y+7)=2$$

$$f(5+y)-f(y+7)=2$$

$$f(y+7)=f(5+y)-2$$

When $y=-3$,

$$f(y+7)=f(5+y)-2$$
$$f(4)=f(2)-2=5-2=3$$

When $y=-1$,

$$f(y+7)=f(5+y)-2$$
$$f(6)=f(4)-2=3-2=1$$

When $y=1$,

$$f(y+7)=f(5+y)-2$$
$$f(8)=f(6)-2=1-2=-1$$

Continuing this method, we discover when y is added 2, the value of function is reduce 2. Hence $f(2018)=f(2011+7)=-1-2010=-2011$.

Answer: -2011

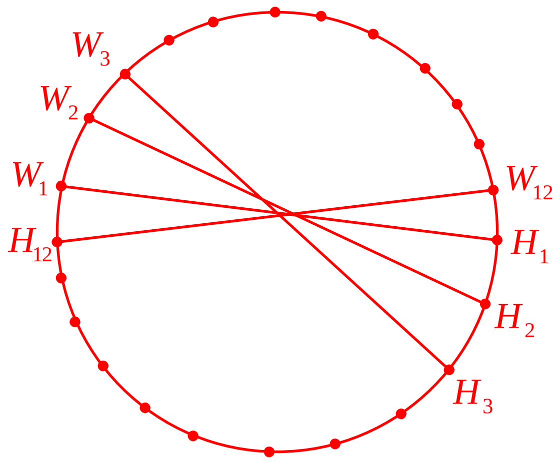
【Marking Scheme】

- Observe $f(y+7)=f(5+y)-2$ when $x=2$, 10 marks.
- Find the value of $f(4)$, 10 marks.
- Observe the rule of the value of function when y is added 2, 10 marks.
- Find the value of $f(2018)$, 10 marks.

3. There are twelve married couples seated around a circular table such that each pair of the husband is seated exactly in front of his respective wife. If we allowed man and woman swap their seat, then what is the minimum number of swapping needed so that each pair of couple is seated next to each other?

【Submitted by *South Africa*】

【Solution】

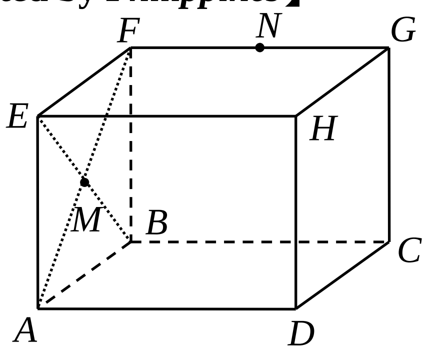


Denote 12 wives as W_1, W_2, W_3, … , W_{12}, and 12 husbands as H_1, H_2, H_3, … , H_{12}.
If we fixed W_1, then H_1 must swap places with W_{12}, then swaps with W_{11}, …, W_2. It need 11 swaps for sitting next to his wife.
Now fixed W_2, then H_2 must swap places with W_{12}, then swaps with W_{11}, … , W_3. It need 10 swaps for sitting next to his wife.
Now fixed W_3, then H_3 must swap places with W_{12}, then swaps with W_{11}, … , W_4. It need 9 swaps for sitting next to his wife. ……..
Now fixed W_{11}, then H_{11} must swap places with W_{12}. It need 1 swap for sitting next to his wife.
In this situation, W_{12} and H_{12} are sitting next to each, no more swap are need.

Hence the total is $11+10+...+1=\dfrac{11\times 12}{2}=66$ swaps.

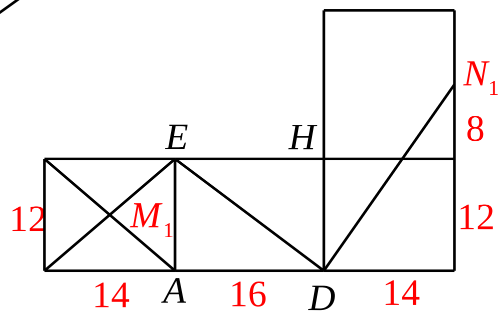
Answer: 66 swaps

4. In the figure shown below, a cuboid *ABCD-EFGH* have side lengths $AE = 12$ cm, $AB = 14$ cm and $AD = 16$ cm. It is known that point M is the centre of the rectangle *ABCD* and point N is the midpoint of the line segment *FG*. It is also known that there is a piece of sugar at both points M and N, and there is an ant at point D. What is the minimum distance, in cm, that the ant must travel in order to get the sugar? 【**Submitted by *Philippines***】



【Solution】

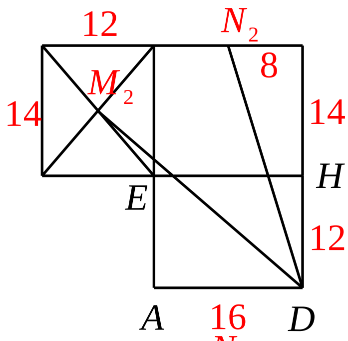
Expand the cuboid along *DH*, *EA*, *EA*, *HG*, and we have the right figure, then



$$DN_1 = \sqrt{14^2 + (8+12)^2} = \sqrt{596}$$

$$DM_1 = \sqrt{6^2 + (16+7)^2} = \sqrt{565}$$

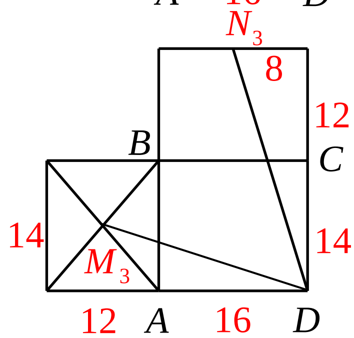
Expand the cuboid along *EH*, *EF*, and we have the right figure, then



$$DN_2 = \sqrt{8^2 + (12+14)^2} = \sqrt{740}$$

$$DM_2 = \sqrt{(12+7)^2 + (16+6)^2} = \sqrt{845}$$

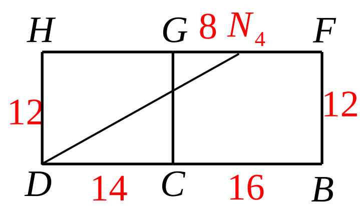
If the ant passes through the bottom surface *ABCD*, as shown in the right figure, then



$$DN_3 = \sqrt{(12+14)^2 + 8^2} = \sqrt{740}$$

$$DM_3 = \sqrt{7^2 + (16+6)^2} = \sqrt{533}$$

Expand the cuboid along *GC*, as shown in the right figure, then



$$DN_4 = \sqrt{(14+8)^2 + 12^2} = \sqrt{628}$$

In summary, ants can find sugar by climbing at least $\sqrt{533}$ units.

Answer: $\sqrt{533}$ cm

【**Marking Scheme**】

- 5 marks for getting correct distance in each situation.
- Conclude the correct answer by using above results, 5 marks.

5. What is the remainder of the sum of the squares of 2018 positive integers when divided by 2018 if the difference between the product of any 2018 of them and the remaining one is divisible by 2018?【**Submitted by *Jury***】

【Solution】

Let the numbers be a_1, a_2, … , a_{2018} and let P be their product. When $1 \le i \le 2018$, 2018 divides $\frac{P}{a_i} - a_i$. Hence it divides $a_i(\frac{P}{a_i} - a_i) = P - a_i^2$. It follows that 2018 divides $P - a_1^2 + P - a_2^2 + ... + P - a_{2018}^2 = 2018P - (a_1^2 + a_2^2 + ... + a_{2018}^2)$.

Hence it divides $a_1^2 + a_2^2 + ... + a_{2018}^2$, i.e. the remainder is 0. *Answer:* 0

6. We know that each of the other 25 students in Peter's class has a different number of friends in the class. What is the minimum possible number of friends Peter has in the class?【**Submitted by *Jury***】

【Solution】

We consider two cases.

Case 1. There is a student with 0 friends.

Then the highest number of friends a student can have is 24. Since the 25 students other than Peter have different numbers of friends, these numbers are 0, 1, …, 24. Put the students with 12 friends or less in Group A and those with 13 or more in Group B, with Peter left out for now. The total number of friends the students in Group A have, counting multiplicity, is $0+1+...+12=78$, and that in Group B is $13+14+...+24=222$. The maximum number of friends the students in Group B have within Group B is $12\times11=132$. This plus 78 is still 12 short of 222. It follows that every student in Group B is a friend of Peter. On the other hand, all the friends of the students in Group A must belong to Group B, so that none of the students in Group A is a friend of Peter. Hence Peter has exactly 12 friends.

Case 2. There are no students with 0 friends.

Then the 25 students other than Peter have 1, 2, …, 25 friends respectively. Let Groups A and B be formed as before, with Peter not in either. As in Case 1, $1+2+...+12=78$, $13+14+...+25=247$, $13\times12=156$ and $247-156-78=13$. Using an analogous argument, we conclude that Peter has exactly 13 friends, all belonging to Group B.

In summary, Peter may have either 12 or 13 friends in the class. The minimal possible values of the number of friends Peter has in the class is 12.

Answer: 12

【**Marking Scheme**】

- In case 1, dividing the students into two groups correctly, 5 marks.
  Find the total number of friends the students in Group A have, 5 marks.
  Find the total number of friends the students in Group B have, 5 marks.
  Conclude Peter has exactly 12 friends, 5 marks.
- In case 2, dividing the students into two groups correctly, 5 marks.
  Find the total number of friends the students in Group A have, 5 marks.
  Find the total number of friends the students in Group B have, 5 marks.
  Conclude Peter has exactly 13 friends, 5 marks.