International Young Mathematicians' Convention

Senior level

Individual Contest

1. If a and b are real numbers such that $a+6b=0$, then what is the remainder when $x^3+\frac{a-b}{a+b}x^2+\frac{2b}{a+b}x+1$ is divided by $x+1$? 【**Submitted by *Philippines***】

【Solution 1】

Since $a+6b=0$, we have $a=-6b$.

$$x^3+\frac{a-b}{a+b}x^2+\frac{2b}{a+b}x+1=x^3+(\frac{-6b-b}{-6b+b})x^2+(\frac{2b}{-6b+b})x+1$$
$$=x^3+\frac{7}{5}x^2-\frac{2}{5}x+1$$

Let $f(x)=x^3+\frac{7}{5}x^2-\frac{2}{5}x+1$. Then $f(-1)=(-1)^3+\frac{7}{5}\times(-1)^2-\frac{2}{5}\times(-1)+1=\frac{9}{5}$.

【Solution 2】

Let $f(x)=x^3+\frac{a-b}{a+b}x^2+\frac{2b}{a+b}x+1$. Then $f(-1)=-1+\frac{a-b}{a+b}-\frac{2b}{a+b}+1=\frac{a-3b}{a+b}$.

Since $a+6b=0$, we have $a=-6b$. Then $f(-1)=\frac{-6b-3b}{-6b+b}=\frac{9}{5}$.

Answer: $\frac{9}{5}$

2. Find the remainder when $2017\times2015\times2013\times\ldots\times1+2018\times2016\times2014\times\ldots\times2$ is divided by 2019. 【**Submitted by *Jury***】

【Solution】

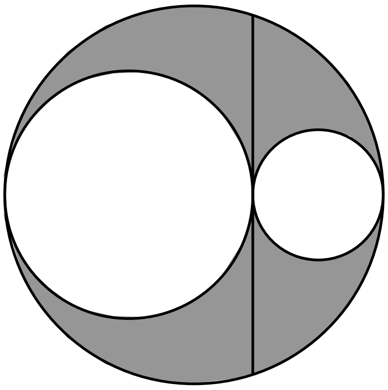
In modulo 2019, we have

$$2017\times2015\times2013\times\ldots\times1+2018\times2016\times2014\times\ldots\times2$$
$$\equiv2017\times2015\times2013\times\ldots\times1+(-1)\times(-3)\times(-5)\times\ldots\times(-2017)$$
$$\equiv(1+(-1)^{\frac{2018}{2}})\times2017\times2015\times2013\times\ldots\times1$$
$$\equiv(1+(-1)^{1009})\times2017\times2015\times2013\times\ldots\times1$$
$$\equiv0$$

Hence the answer is 0.

Answer: 0

3. The length of the chord of a circle is 2 cm. The two smaller circles are tangent to the large circle and also to each other at the midpoint of the chord. Find the area, in cm^2, of the shaded region. 【**Submitted by *Jury***】

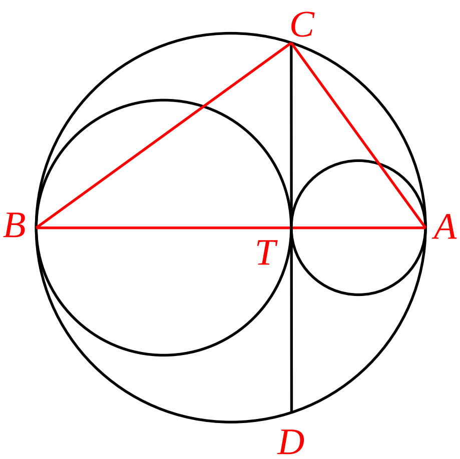


【Solution 1】

Let the chord be *CD* with midpoint *T*. Let *AB* be the diameter of the large circle through *T*. Then it is perpendicular to *CD*. Moreover, *AT* and *BT* are diameters of the small circles respectively. Let $AT=2x$ and $BT=2y$. Now $\angle ACB=90°$. By Pythagoras' Theorem,

$$4x^2+4y^2+8xy=AB^2=AC^2+BC^2=AT^2+BT^2+2CT^2=4x^2+4y^2+2.$$

Hence, $xy=\dfrac{1}{4}$. The area of the region inside the large circle but outside the small circles is given by $\pi(x+y)^2-\pi x^2-\pi y^2=2\pi xy=\dfrac{\pi}{2}$ cm^2.



【Solution 2】

Let the chord be *CD* with midpoint *T*. Let *AB* be the diameter of the large circle through *T*. Let $AT=2x$ and $BT=2y$.

By Intersecting Chords theorem, $AT\times BT=CT\times DT$, i.e. $4xy=1$. Hence $xy=\dfrac{1}{4}$.

The area of the region inside the large circle but outside the small circles is given by

$\pi(x+y)^2-\pi x^2-\pi y^2=2\pi xy=\dfrac{\pi}{2}$ cm^2.

Answer: $\dfrac{\pi}{2}$ cm^2

4. Find the sum of the roots of the equation :

$$\sqrt{3x^2+x-1}+\sqrt{x^2-2x-3}=\sqrt{3x^2+3x+5}+\sqrt{x^2+3}.$$ 【**Submitted by *India***】

【Solution】

Let $u=\sqrt{3x^2+x-1}$, $v=\sqrt{x^2-2x-3}$, $w=\sqrt{3x^2+3x+5}$ and $z=\sqrt{x^2+3}$.

Then $u+v=w+z$ and

$u^2-v^2=(3x^2+x-1)-(x^2-2x-3)=2x^2+3x+2=w^2-z^2$, yields $u-v=w-z$.

Therefore $u=w$. Then $\sqrt{3x^2+x-1}=\sqrt{3x^2+3x+5}$, i.e. $x-1=3x+5$. So $x=-3$.

Answer: -3

5. There are seven tokens having different weights namely 1g, 2g, 4g, 8g, 16g, 32g and 64g. How many different ways can we get a weight of 21g using a regular two-sided weighing scale? (Note: Each token may be placed on either pan of the balance, and it is not necessary to use all the tokens in each weigh). 【**Submitted by *Jury***】

【Solution】

Case (i) the token of weight 64 g is not used:

If the token of weight 32 g is not used, then the number of ways is clearly 5:

$$21=16+4+1=16+8-2-1=16+4+2-1=16+8-4+1=16+8-4+2-1.$$

If32 g is used, it must be placed on the opposite pan. The number of ways is then same as the number of ways of balancing an object of weight $32-21=11$ g by using a set of tokens having weights 1 g, 2 g, 4 g, 8 g and 16 g. Observe that

$$11=8+2+1=8+4-1=16-8+2+1=16-8+4-1$$
$$=16-4-1=16-4-2+1=8+4-2+1=16-8+4-2+1$$

The number of ways is then 8. Hence balancing an object of weight 21 g is $5+8=13$ ways.

Case (ii) the token of weight 64 g is used:

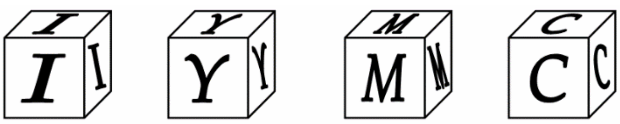
It must be placed on the opposite pan. The number of ways is then same as the number of ways of balancing an object of weight $64-21=43$ g by using a set of tokens having weights 1 g, 2 g, 4 g, 8 g, 16 g and 32 g.

Since $1+2+4+8+16=31<43$, the token of weight 32 g must be used and placed on the same pan as object. The number of ways is then same as the number of ways of balancing an object of weight $43-32=11$ g by using a set of tokens having weights 1 g, 2 g, 4 g, 8 g and 16 g. From case (i), there are 8 ways.

So balancing an object of weight 21 g is $13+8=21$ ways.

Answer: 21 ways

6. There are 8 wooden blocks, two of these wooden blocks have the letter "I" written on each face, another two wooden blocks have the letter "Y" written on each face, another two wooden blocks have the letter "M" written on each face and then last two wooden blocks have the letter "C" written on each face. What is the probability that when we take four out of the eight wooden blocks we can spell out the word "IYMC"? 【**Submitted by *Philippines***】



【Solution 1】

In general, when 1 block is taken out from 8 wood blocks, there are 8 ways to select one block; a 2nd block is taken out from the remaining 7 wooden blocks, then there are 7 ways of selecting; follow a 3rd block is selected from the remaining 6 wooden blocks, there are 6 ways; finally the 4th block is taken out from the remaining 5 wooden blocks, so there are 5 ways. Hence, taken out 4 blocks from 8 wood blocks, and there are $8\times7\times6\times5=1670$ ways.

When select four blocks which contain one "I", one "Y", one "M" and one "C".

To select the first block, there will be 8 different ways. To select the second block whose letter appear must not the same as the first block, there will be 6 different ways, to select the third block whose letter must totally different with the first two blocks, there will be 4 different ways and finally to select the fourth block whose letter must distinct with the first three block, so there will be 2 different ways. Thus, there are $8\times6\times4\times2=384$ ways.

Therefore, the probability of selecting four blocks from 8 wooden block that can be arrange as IYMC is $\frac{384}{1670}=\frac{8}{35}$.

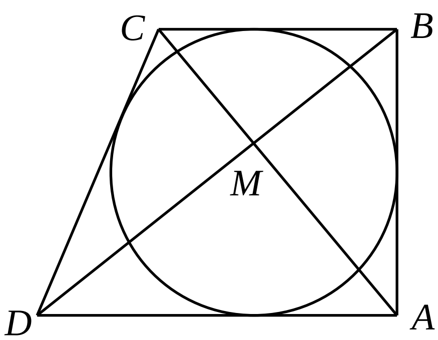
【Solution 2】

When we take out 4 blocks from 8 wood blocks in a same time, there are $C_4^8=\frac{8\times7\times6\times5}{4\times3\times2\times1}=70$ ways to select. There are 2 ways to select one "I", 2 ways to select one "Y", 2 ways to select one "M" and 2 ways to select one "C".

Hence the probability of selecting four blocks from 8 wooden block that can be arrange as IYMC is $\frac{16}{70}=\frac{8}{35}$.

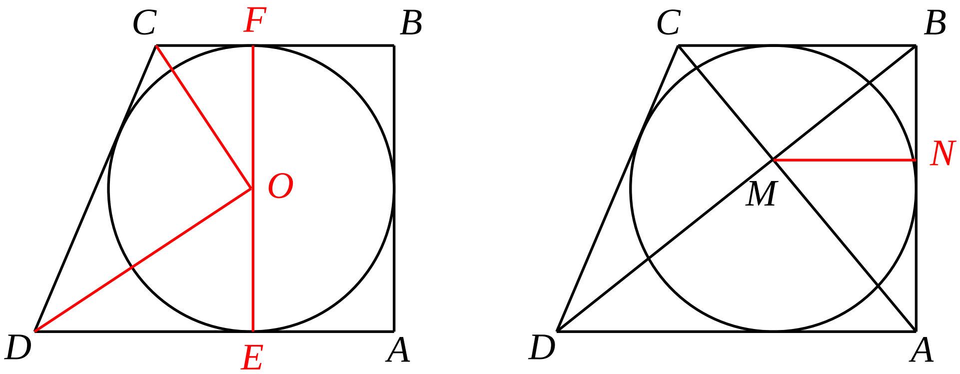
Answer: $\frac{8}{35}$

7. In the figure, *ABCD* is a quadrilateral that has an incircle with radius 10 cm. Side *AD* is parallel to *BC* and perpendicular to *AB* and point *M* is intersection between *AC* and *BD*. Determine the area, in cm^2, of triangle *DCM*. 【**Submitted by *Jury***】



【Solution】

In the diagram below on the left, *O* is the incentre and *EF* is the diameter parallel to *AB*. It is easy to see that triangles *OCF* and *ODE* are similar. From $\frac{CF}{OF}=\frac{OE}{DE}$, we have $CF \times DE = 10^2 = 100$.



In the diagram above on the right, MN is perpendicular to AB. Then triangles AMN and ACB are similar. From $\frac{MN}{BC} = \frac{AN}{AB}$, we have $\frac{MN}{10+CF} = \frac{AN}{20}$. Similarly, $\frac{MN}{10+DE} = \frac{20-AN}{20}$. Addition yields $MN(\frac{1}{10+CF} + \frac{1}{10+DE}) = 1$.

Hence $MN = \frac{(10+CF)(10+DE)}{20+CF+DE}$.

Since $CF \times DE = 100$, $(10+CF)(10+DE) = 10(20+CF+DE)$. Hence, $MN = 10\,\text{cm}$ so that the area of triangle ABM is $10^2 = 100\,\text{cm}^2$. Since BC is parallel to AD, triangles ABD and ACD have equal area. Hence triangles ABM and CDM have equal area too, so that the area of triangle CDM is 100 cm^2.

Answer: 100 cm^2

8. If $a^6 + b^6 + c^6 + d^6 + e^6 + f^6 - 1 = 6abcdef$, where a, b, c, d, e and f are integers. How many possible values are there for $a+b+c+d+e+f$? 【**Submitted by *Jury***】

【Solution】

By the Arithmetic-Geometric Means Inequality,

$$6abcdef = a^6 + b^6 + c^6 + d^6 + e^6 + f^6 - 1 \geq 3a^2b^2c^2 + 3d^2e^2f^2 - 1.$$

This is equivalent to $1 \geq 3(abc - def)^2$. Since a, b, c, d, e and f are integers, we must have $abc = def$. By symmetry, the product of any three of them is equal to that of the other three. If none of them is 0 then they must all be equal. However $6a^6 = 6a^6 - 1$ cannot then hold. Hence, at least one of them is 0, so that $6abcdef = 0$. From $a^6 + b^6 + c^6 + d^6 + e^6 + f^6 = 1$, we must have five of them equal to 0 and the other equal to 1 or -1. So there are 2 possible values.

Answer: 2