The Eighth International Young Mathematicians' Convention

IYMC-Mathematica 2018

2nd to 5th December 2018

Organised by: City Montessori School, Gomti Nagar Campus-1, Lucknow, India

International Young Mathematicians' Convention

Senior level

Team Contest

1. Let a and b be real numbers such that the equation $x^4+ax^3+2x^2+bx+1=0$ has at least one real root, what is the minimum possible value of a^2+b^2?
   【**Submitted by *Jury***】

【Solution】

Let r be a real root of the equation. We then write the equation as $ar^2+b=\frac{(r^2+1)^2}{r}$.

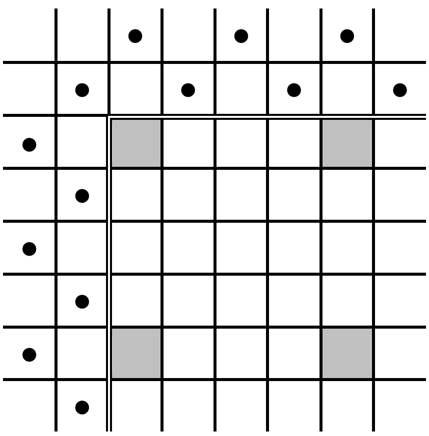
By Cauchy's Inequality, $(r^4+1)(a^2+b^2)\geq(ar^2+b)^2$. By the Arithmetic-Geometric Means Inequality, we have

$$\begin{aligned}a^2+b^2&\geq\frac{(r^2+1)^4}{r^2(r^4+1)}\\&=\frac{r^8+2r^4+1+4r^4+4r^6+4r^2}{r^2(r^4+1)}\\&=\frac{r^4+1}{r^2}+\frac{4r^2}{(r^4+1)}+4\\&\geq4+4=8\end{aligned}$$

Answer: 8

2. On an infinite chessboard, the squares at the intersections of every fourth row and every fourth column are removed. Prove that it is not possible for a Knight to visit every square exactly once which has not been removed. 【**Submitted by *Jury***】

【Solution】



Consider a 61×61 subboard with the four corners squares removed, among others. These squares are shaded in the diagram below which shows the upper left corner of this subboard, bounded by the double lines. If we paint the infinite chessboard in the usual pattern, all shaded squares have the same colour, say black.

Now the subboard has $61^2=3721$ squares, $\frac{3721-1}{2}=1860$ of which are white. Of the 1861 black squares, $16^2=256$ have been removed, leaving behind $1861-256=1605$. There are also $4\times(61+1)=248$ black squares outside the subboard that are within a Knight's move from some white squares inside the subboard. These are marked with black dots in the diagram. Since $1605+248=1853<1860$, it is impossible for a Knight to tour every white square in the subboard because a Knight must visit squares of opposite colours in two consecutive moves.

【Marking Scheme】

- Consider a 61×61 subboard with the four corners squares removed, among others, 10 marks.
- Observe there are 1860 white squares, 5 marks.
- Observe there are 1605 black squares without being removed, 10 marks.
- Observe there are 248 black squares outside the subboard that are within a Knight's move from some white squares inside the subboard, 10 marks.
- Observe the result holds since $1605+248=1853<1860$, 5 marks.

3. Suppose $x=\dfrac{a}{a^2+16}$, when a is a real number. What is the minimum value of $\sqrt{1+8x}+\sqrt{1-8x}$? 【**Submitted by *Philippines***】

【Solution】

$$\begin{aligned}\sqrt{1+8x}+\sqrt{1-8x}&=\sqrt{1+\frac{8a}{a^2+16}}+\sqrt{1-\frac{8a}{a^2+16}}\\&=\sqrt{\frac{a^2+8a+16}{a^2+16}}+\sqrt{\frac{a^2-8a+16}{a^2+16}}\\&=\sqrt{\frac{(a+4)^2}{a^2+16}}+\sqrt{\frac{(a-4)^2}{a^2+16}}\\&=\frac{|a+4|+|a-4|}{\sqrt{a^2+16}}\end{aligned}$$

Case 1: When $a<-4$:

Above expression is equal to $\dfrac{-a-4-a+4}{\sqrt{a^2+16}}=-\dfrac{2a\sqrt{a^2+16}}{a^2+16}>\dfrac{8\sqrt{32}}{32}=\sqrt{2}$;

Case 2: When $-4\le a<4$:

Above expression is equal to $\dfrac{a+4-a+4}{\sqrt{a^2+16}}=\dfrac{8\sqrt{a^2+16}}{a^2+16}\ge\dfrac{8\sqrt{16}}{16}=2$. In this case, the minimum value happens when $a=0$.

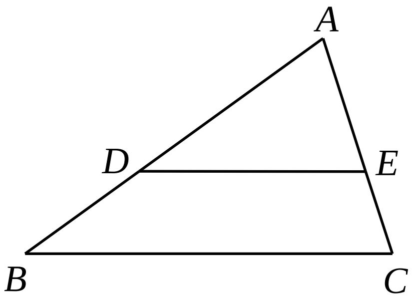
Case 3: When $a\ge 4$:

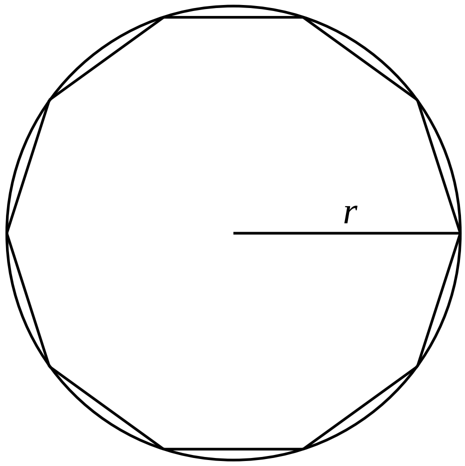
Above expression is equal to $\dfrac{a+4+a-4}{\sqrt{a^2+16}}=\dfrac{2a\sqrt{a^2+16}}{a^2+16}\ge\dfrac{8\sqrt{32}}{32}=\sqrt{2}$.

Thus, the minimum value of $\sqrt{1+8x}+\sqrt{1-8x}$ is $\sqrt{2}$.

Answer: $\sqrt{2}$

4. In the figure, points *D* and *E* lies along sides *AB* and *AC* of triangle *ABC* such that *DE* is parallel to *BC*. It is known that $AD = DE = AC = r$ and $BD = AE = s$. Now, a regular decagon is inscribed in a circle whose radius is *r*. Prove that the length to a side of this decagon is equal to *s*. 【**Submitted by *Jury***】

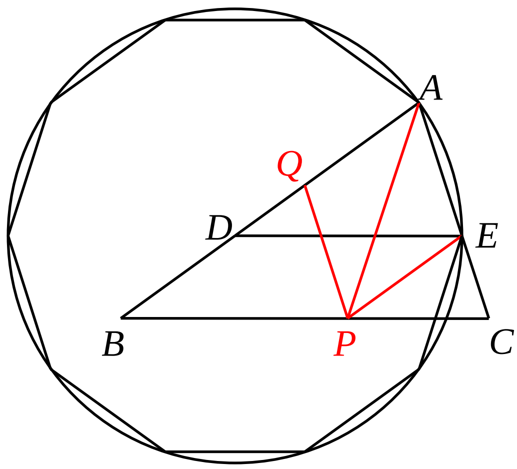




【Solution 1】

Since $AD = DE = AC$, the circle with radius *AC* may be centered at *D* and passing through *A* and *E*. Since $BD = AE$, the problem is equivalent to proving that $\angle ADE = 36°$.

Since $AD = DE$ and *DE* is parallel to *BC*, $\angle DAE = \angle DEA = \angle BCA$, so that $AB = BC$. Complete the parallelograms *BDEP* and *AEPQ*. Then $AE = BD = PE$, so that *AEPQ* is actually a rhombus. Let $\angle PAQ = \theta$. Then $\angle PAE = \theta$ and it follows that $\angle BQP = \angle ACP = 2\theta$. Since $BQ = BP = DE = AC$ and $PQ = AQ = PC$, triangles *BQP* and *ACP* are congruent, so that $\angle ADE = \angle QBP = \angle CAP = \theta$. It follows that $5\theta = 180°$ and indeed $\theta = 36°$.



【**Marking Scheme**】

- Observe the problem is equivalent to proving that $\angle ADE = 36°$, 5 marks.
- Show that $AB = BC$, 10 marks.
- Construct the parallelograms *BDEP* and *AEPQ*, and then show that *AEPQ* is a rhombus, 10 marks.
- Show that triangles *BQP* and *ACP* are congruent, 10 marks.
- Conclude that $\angle ADE = 36°$, 5 marks.

【Solution 2】

Since triangles ADE and ABC are similar, then $\frac{AD}{AB}=\frac{AE}{AC}=\frac{DE}{BC}=\frac{r}{r+s}=\frac{s}{r}$.

We have $r^2=rs+s^2$, that is $s^2+rs-r^2=0$. Solve the equation, $s=\frac{-1\pm\sqrt{5}}{2}r$.

s can not be negative. Hence $s=\frac{-1+\sqrt{5}}{2}r=\varphi r$.

On other hand, a regular decagon is inscribed in a circle whose radius is r, the length to a side of this decagon is equal to $r\varphi$. Hence it length is s.

【Marking Scheme】

- Show that triangles ADE and ABC are similar and $\frac{AD}{AB}=\frac{AE}{AC}=\frac{DE}{BC}$, 10 marks.
- Show that $s=\frac{-1+\sqrt{5}}{2}r=\varphi r$ 10 marks.
- Show that the length to a side of this decagon is equal to $r\varphi$, 20 marks.

5. For any positive integer n and non-zero digits a, b and c, let A_n be an n-digit integer each of whose digits is equal to a; let B_n be an n-digit integer each of whose digits is equal to b and let C_n be an $2n$-digit (not n-digit) integer each of whose digits is equal to c. What is the maximum value of $a+b+c$ for which there are at least two values of n such that $C_n-B_n=A_n^2$? **【Submitted by *Thailand*】**

【Solution】

Observe $A_n=a(1+10+10^2+\cdots+10^{n-1})=a\times\frac{10^n-1}{9}$; similarly $B_n=b\times\frac{10^n-1}{9}$ and $C_n=c\times\frac{10^{2n}-1}{9}$. The relation $C_n-B_n=A_n^2$ can be rewritten as

$$c\times\frac{10^{2n}-1}{9}-b\times\frac{10^n-1}{9}=a^2\times(\frac{10^n-1}{9})^2.$$

Since $n>0$, $10^n>1$ and we may cancel out a factor of $\frac{10^n-1}{9}$ to obtain

$$c\times(10^n+1)-b=a^2\times(\frac{10^n-1}{9}).$$

This is a linear equation in 10^n. Thus, if two distinct values of n satisfy it, then all values of n will. Matching coefficients, we get

$$c=\frac{a^2}{9} \text{ and } c-b=-\frac{a^2}{9}, \text{ so } b=\frac{2a^2}{9}.$$

To maximize $a+b+c=a+\frac{a^2}{3}$, we need to maximize a. Since b and c must be integers, a must be a multiple of 3. If $a=9$, then b exceeds 9. However, if $a=6$, then $b=8$ and $c=4$ for an answer of 18.

Answer: 18

6. How many positive integers $n \le 2018$ are there so that it is possible to arrange the numbers from 1 to n in some order, such that the average of any group of two or more adjacent numbers is not an integer? 【**Submitted by *Jury***】

【Solution】

The sum of n consecutive numbers is $\frac{n(2a+n-1)}{2}$ where a is the first of these numbers. Their average is $\frac{2a+n-1}{2}$, which is an integer if and only if n is odd. In our problem, n cannot be odd. We now show that n can be any even number. Arrange the n numbers in their natural order and group them into pairs. Reverse the order within each pair to yield the arrangement 2, 1, 4, 3, 6, 5, … , n, $n-1$. Consider any k where $2 \le k \le n$. Consider first the case where k is odd. Any k adjacent numbers in our arrangement consist of k consecutive integers except that the one which is not in a pair is replaced by its partner, which differs from it by 1. Thus the sum of these k numbers is $mk \pm 1$ for some m, so that their average is not an integer. Finally, consider the case where k is even. Any k adjacent numbers in our arrangement consist of k consecutive integers, possibly with the two at the ends not being in pairs and replaced by their partners. Since one would be increased by 1 while the other would be decreased by 1, the sum is not affected by the replacement. So the average is not an integer. Thus there are $\frac{2018}{2}=1009$ positive integers.

Answer: 1009

【Marking Scheme】

- Observe n can not be odd, 10 marks.
- Suppose n is even and then yield the arrangement 2, 1, 4, 3, 6, 5, …, n, $n-1$, 5 marks.
- Consider any k where $2 \le k \le n$.
  Show that their average is not an integer as k is odd, 10 marks.
  Show that their average is not an integer as k is even, 10 marks.
- Conclude that n is even and the answer is 1009, 5 marks.