Key Stage 3 - Individual Contest

Section A.

In this section, there are 12 questions. Fill in the correct answer in the space provided at the end of each question. Each correct answer is worth 5 points.

1. If both a and b are positive integers greater than 1, find the smallest possible sum of a and b, such that $\sqrt{a\sqrt{a}}=b$. 【**Submitted by *Bulgaria***】

【Solution】

$\sqrt{a\sqrt{a}}=b \Rightarrow a\sqrt{a}=b^2 \Rightarrow a^3=b^4$. Then b is a perfect cube. When $b=2^3=8$, we get $a^3=\left(2^3\right)^4 \Rightarrow a=16$. So, $a+b=16+8=24$.

Answer: 24

2. Find the largest positive integer d, in which there exists at least one integer n such that d divides both n^2+1 and $(n+1)^2+1$. 【**Submitted by *Sri Lanka***】

【Solution】

Let $d\,|\,n^2+1$ and $d\,|\,(n+1)^2+1$, or $d\,|\,n^2+2n+2$.

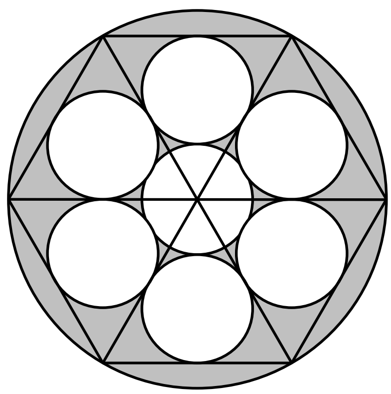
Then, $d\,|\,(n^2+2n+2)-(n^2+1)$, or $d\,|\,2n+1 \Rightarrow d\,|\,4n^2+4n+1$,

So, $d\,|\,4(n^2+2n+2)-(4n^2+4n+1)$, or $d\,|\,4n+7$.

Then, $d\,|\,(4n+7)-2(2n+1)$, or $d\,|\,5$, so d can only be 1 or 5. The largest d is 5.

Answer: 5

3. A regular hexagon is inscribed in a circle with radius 12 cm. The hexagon is divided into 6 congruent equilateral triangles and each of them has a small circle inscribed in it. Another small circle is then drawn touching all the inscribed circles. What is the area of the shaded region? Take $\pi=3.14$. 【**Submitted by *Bulgaria_FPMG***】



【Solution】

The radius of smaller circle is $r=\frac{1}{3}\times\frac{12\sqrt{3}}{2}=2\sqrt{3}$ cm.

The area is $S_{gc}-7\times S_{sc}=12^2\pi-7\times\left(2\sqrt{3}\right)^2\pi=(144-84)\pi=188.4$ cm^2.

Answer: 188.4 cm^2

4. Find the smallest positive integer *n* so that 2*n* is a perfect square and 7*n* is a perfect 7^{th} power. 【**Submitted by *Thailand***】

【Solution】

Let $2n=a^2$ and $7n=b^7$, where *a* and *b* are positive integers.
a is divisible by 2 $\rightarrow a^2=2n$ is divisible by 2^2 and *n* is divisible by 2.
The same way *n* is divisible by 7.
Since we want to find the smallest positive integer n, we can let $n=2^p7^q$ where *p* and *q* are positive integers.
Then, $2n=2^{p+1}7^q=a^2$, $7n=2^p7^{q+1}=b^7$.
If $2^{p+1}7^q$ is a perfect square, then $p+1$ and *q* must be divisible by 2.
Let $p+1=2p_1$, $q=2q_1$, where p_1 and q_1 are positive integers.
If 2^p7^{q+1} is a perfect 7^{th} power, then *p* and $q+1$ must be divisible by 7 and *r* must be perfect 7^{th} power. Thus, $p=2p_1-1=7s$ and $q+1=2q_1+1=7t$.
By inspection, we see that the smallest integer p_1 for which $2p_1-1$ is divisible by 7 is $p_1=4$, $p=2\times4-1=7$.
So, the smallest integer q_1 in which $2q_1+1$ is divisible by 7 is $q_1=3$,
$q=2\times3=6$.
Therefore, the smallest positive integer *n* is $n=2^77^6$.

Answer: 2^77^6

5. Find all possible integer values of *x*, in which $\left(\frac{21}{x}-2\right)^2-2\left(\frac{21}{x}-2\right)=x+42$.

【**Submitted by *Bulgaria***】

【Solution】

$$\left(\frac{21}{x}-2\right)^2-2\left(\frac{21}{x}-2\right)+1=x+43\Leftrightarrow\left(\left(\frac{21}{x}-2\right)-1\right)^2=x+43\text{, i.e. }\left(\frac{21}{x}-3\right)^2=x+43.$$

Since *x* is an integer, its left side accepts only whole values. Therefore, *x* divides 21, i.e. $x=\pm1;\ \pm3;\ \pm7;\ \pm21$. When $x=-7$ and $x=21$ the number $x+43$ is a perfect square. It is easy to check that -7 is a solution, but 21 is not a solution. So, $x=-7$.

Answer: -7

6. If the equation $\left(a^2+3b^2\right)x^2-(4a+6b)x+7=0$ has a root of 2017, where *a* and *b* are real numbers, find the sum of *a* and *b*. 【**Submitted by *Bulgaria***】

【Solution】

If $a=b=0$, we get an equation, which has no solutions. So, at least one of the numbers a, b is different from zero and then the equation is quadratic. Discriminant $D=\left(4a+6b\right)^2-28\left(a^2+3b^2\right)=-12\left(a^2-4ab+4b^2\right)=-12\left(a-2b\right)^2\le 0$ for any a and b. On the other side, given equation has a real root when $D\ge 0$. It is possible only when $D=0$, from where 2017 is the only solution of equation and $a=2b$. Therefore, $2017=\frac{4a+6b}{2(a^2+3b^2)}=\frac{14b}{14b^2}=\frac{1}{b}$, and $b=\frac{1}{2017}$ and $a=\frac{2}{2017}$.

Answer: $\frac{3}{2017}$

7. If $x>1$, find all possible values of x that satisfies the following equation: $\frac{x-2017}{2018}-\frac{x-2018}{2017}=\frac{2017}{x-2018}-\frac{2018}{x-2017}$. 【**Submitted by *Bulgaria***】

【Solution】

Substitute $y=\frac{x-2017}{2018}$ and $z=\frac{x-2018}{2017}$. Then $y-z=\frac{1}{z}-\frac{1}{y}$, i.e. $(y-z)(yz-1)=0$. Therefore,

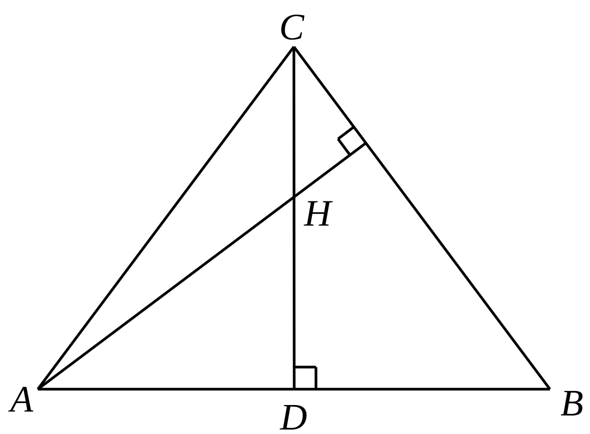
(1) $y=z$, i.e. $\frac{x-2017}{2018}=\frac{x-2018}{2017}$, and $x=4035$.

(2) $yz=1$, i.e. $\frac{x-2017}{2018}\times\frac{x-2018}{2017}=1$, and $x^2-4035x=0$.

Since $x>1$, $x=4035$.

Answer: 4035.

8. In the figure below, *ABC* is an isosceles triangle with base *AB*. Its orthocentre *H* divides its altitude *CD* into two segments *CH* and *HD*, where $CH=7$ cm and $HD=9$ cm. Find the perimeter of triangle *ABC*, in cm. 【**Submitted by *Bulgaria_SMG***】



【Solution】

If CD intersects the circumcircle of $\triangle ABC$ at point N, then $\triangle ADH \cong \triangle ADN$.

So $DN = DH = 9$.

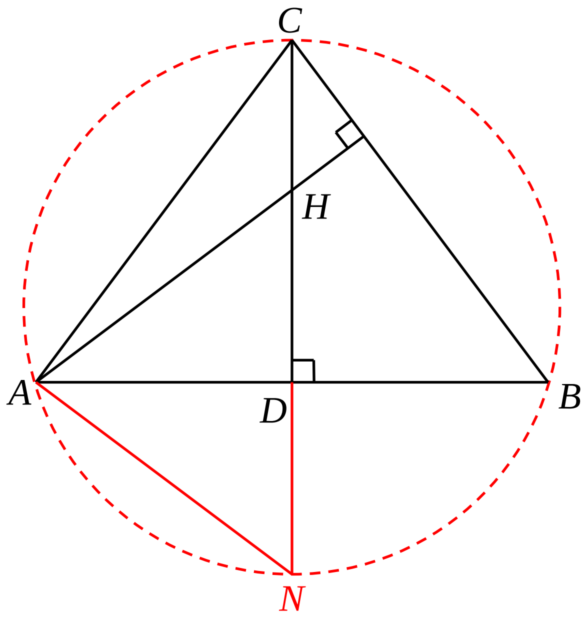
For chords AB and CN,

$AD \times DB = CD \times DN \Rightarrow (AD)^2 = 16 \times 9$

$\Rightarrow AD = 12$

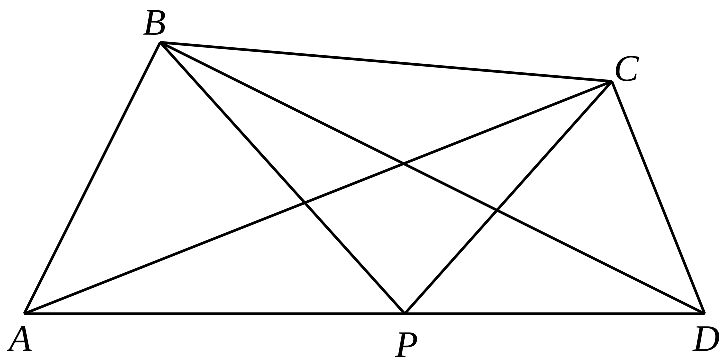
$AC^2 = AD^2 + CD^2 = 12^2 + 16^2 \Rightarrow AC = 20$.

Hence, the perimeter of $\triangle ABC$ is $20 + 20 + 24 = 64$ cm.



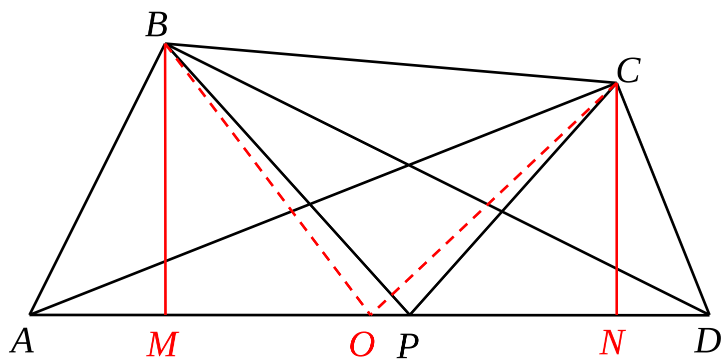
Answer: 64 cm.

9. In quadrilateral $ABCD$, $\angle ABD = \angle ACD = 90°$ and a point P is on AD so that $\angle APB = \angle CPD$, as shown in the figure below. If $AP = 24$ cm and $DP = 19$ cm, find the value of $PB \times PC$. 【**Submitted by *Thailand***】



【Solution 1】

Let M and N be on AD so that $BM \perp AD$ and $CN \perp AD$. Let O be the midpoint of AD. Connect OB and OC.



Since OB and OC are medians of the hypotenuse of the right triangles ABD and ACD, respectively, $OB = OC = OA = OD = \frac{1}{2}AD$. Now let $AM = y$ and $DN = x$, then from $OB^2 - OM^2 = BM^2 = PB^2 - PM^2$, we have

$$OB^2 - (OA - AM)^2 = PB^2 - (PA - AM)^2$$

$$OB^2 - OA^2 + 2\times OA\times AM - AM^2 = PB^2 - PA^2 + 2\times PA\times AM - AM^2$$

$$2\times OA\times AM = PB^2 - PA^2 + 2\times PA\times AM$$

$$PA^2 - PB^2 = 2\times PA\times AM - 2\times OA\times AM$$

$$24^2 - PB^2 = 5y$$

$$PB^2 = 576 - 5y$$

Similarly, we also have $PC^2 = 5x + 361$ from $OC^2 - ON^2 = PC^2 - PN^2$.

Since $\angle MPB = \angle NPC$ and $\angle BMP = 90° = \angle CNP$, triangles *BMP* and *CNP* are similar. Hence, $\frac{PB}{PC} = \frac{PM}{PN} = \frac{AP - AM}{CP - CN} = \frac{24 - y}{19 - x} = \frac{120 - 5y}{95 - 5x}$.

Observe that $120 - 5y = PB^2 - 456$ and $95 - 5x = 456 - PC^2$, so $\frac{PB}{PC} = \frac{PB^2 - 456}{456 - PC^2}$.

Hence, $PB(456 - PC^2) = PC(PB^2 - 456)$, i.e. $(PB + PC)PB\times PC = 456(PB + PC)$.

Thus, $PB\times PC = 456$.

【Solution 2】

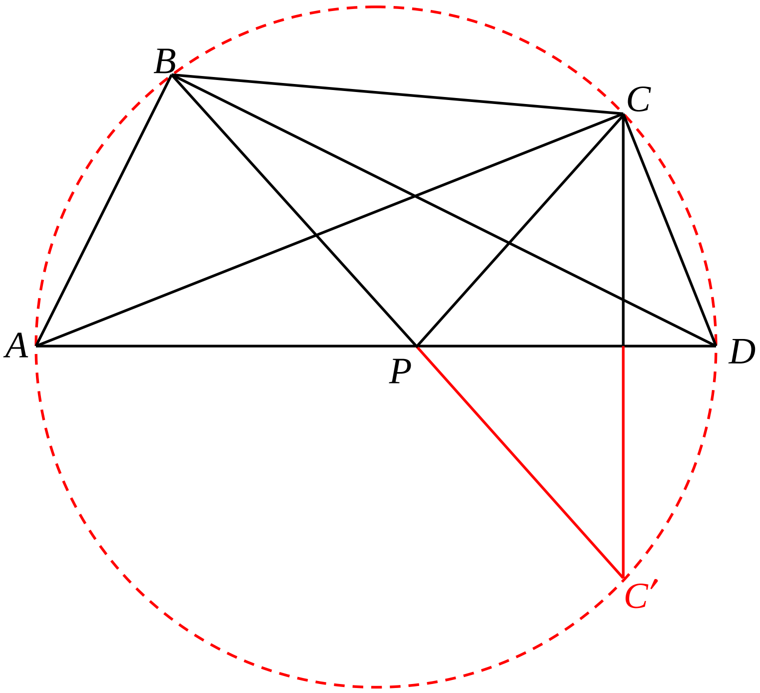
Since $\angle ABD = \angle ACD = 90°$, we know that points *A*, *B*, *C* and *D* are concyclic.

Let point C' be symmetric point of *C* along *AD*, as shown in the figure below.

Then, we have $\angle C'PD = \angle CPD = \angle APB$. Hence, points *B*, *P* and *C'* are collinear.

Using the Intersecting Chord Theorem, we get

$$BP\times PC = BP\times PC' = AP\times PD = 24\times 19 = 456.$$



Answer: 456

10. A girl tosses a fair coin 100 times and a boy tosses a fair coin 101 times. The boy wins if he has more heads than the girl has, otherwise, he loses. Find the probability that the boy win this game. 【**Submitted by *Central Jury***】

【Solution】

Let us pause the game just before the boy makes his last toss. At this point, both players has made 100 tosses or a total of 200 tosses in all. If the girl got more heads than the boy did, the last toss wouldn't help him. If he got more heads than the girl did, the last toss is unnecessary. By symmetry, these two scenarios are equally likely. The last toss only matters if both have the same number of heads at this point. On the last toss, the boy wins if he gets a head, and loses if he gets a tail. Since the coin is fair, either is equally likely. Overall, there is a 50% probability for the boy to win the game.

Answer: 50%

11. A 6×6 chessboard is formed by using 36 unit squares. How many different combinations of 4 unit squares can be selected from the chessboard so that no two unit squares are in the same row or column? 【**Submitted by *Thailand***】

【Solution】

After one of the 36 blocks is chosen, 25 of the remaining blocks do not share its row or column. After the second block is chosen, 16 of the remaining blocks do not share a row or column with either of the first two, 9 of the remaining blocks do not share a row or column with either of the first two. Because the four blocks can be chosen in any order, the number of different combinations is $\dfrac{36\times25\times16\times9}{4!}=5400$.

12. If $\overline{abcd}$ is a 4-digit number, where each different letter represents a different digit such that $a<b$, $c<b$ and $c<d$. How many such 4-digit numbers are there? 【**Submitted by *Bulgaria_FPMG***】

【Solution】

Observe that $a\neq0$ since *a* is the leading digit of a four-digit number.

If $b=2$, then $a=1$, $c=0$ and hence $3\le d\le9$. There are 7 such numbers.

If $b=3$, then $a=1$ or 2.

As $a=1$, then $c=0$ or 2. When $c=0$, then $d=2$ or $4\le d\le9$. There are 7 such numbers. When $c=2$, then $4\le d\le9$. There are 6 such numbers.

As $a=2$, then $c=0$ or 1. When $c=0$, then $d=1$ or $4\le d\le9$. There are 7 such numbers. When $c=1$, then $4\le d\le9$. There are 6 such numbers.

Hence there are totally $2\times(7+6)=26$ such numbers.

If $b=4$, then there are totally $3\times(7+6+5)=54$ numbers by similar argument.

If $b=5$, there are totally $4\times(7+6+5+4)=88$ numbers by similar argument.

If $b=6$, there are totally $5\times(7+6+5+4+3)=125$ numbers by similar argument.

If $b=7$, there are totally $6\times(7+6+5+4+3+2)=162$ numbers by similar argument.

If $b=8$, there are totally $7\times(7+6+5+4+3+2+1)=196$ numbers by similar argument.

If $b=9$, there are totally $8\times(7+6+5+4+3+2+1)=224$ numbers by similar argument.

So, there is a total of $7+26+54+88+125+162+196+224=882$ numbers.

Answer: 882

Section B.

Answer the following 3 questions, each question is worth 20 points. Partial credits may be awarded. Show your detailed solution in the space provided.

1. Let α, β and γ be the three roots of the polynomial x^3-5x+1. If p and q are relatively prime positive integers such that

$$-\frac{p}{q}=\frac{\alpha^3}{(6\beta+1)(6\gamma+1)}+\frac{\beta^3}{(6\alpha+1)(6\gamma+1)}+\frac{\gamma^3}{(6\beta+1)(6\alpha+1)}.$$

Find the sum of p and q. 【**Submitted by *Bulgaria_FPMG***】

【Solution】

Note that $(x-\alpha)(x-\beta)(x-\gamma)=x^3-5x+1 \Rightarrow \alpha+\beta+\gamma=0$, $\alpha\beta+\beta\gamma+\gamma\alpha=-5$ and $\alpha\times\beta\times\gamma=-1$.

Also, it follows that,

$0=(\alpha+\beta+\gamma)^2=\alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\alpha\gamma)\Rightarrow\alpha^2+\beta^2+\gamma^2=10$. Now

$$\begin{aligned}-\frac{p}{q}&=\frac{\alpha^3}{(6\beta+1)(6\gamma+1)}+\frac{\beta^3}{(6\alpha+1)(6\gamma+1)}+\frac{\gamma^3}{(6\beta+1)(6\alpha+1)}\\&=\frac{(5\alpha-1)(6\alpha+1)+(5\beta-1)(6\beta+1)+(5\gamma-1)(6\gamma+1)}{(6\alpha+1)(6\beta+1)(6\gamma+1)}\\&=\frac{30(\alpha^2+\beta^2+\gamma^2)-(\alpha+\beta+\gamma)-3}{216\alpha\beta\gamma+36(\alpha\beta+\beta\gamma+\alpha\gamma)+6(\alpha+\beta+\gamma)+1}\\&=-\frac{297}{395}\end{aligned}$$

Hence, $p+q=297+395=692$.

Answer: 692

【Marking Scheme】

- Full solution with correct answer, 20 marks.
- Correct answer only, 10 marks.
- Finding $\begin{cases}\alpha+\beta+\gamma=0\\ \alpha\beta+\alpha\gamma+\gamma\alpha=-5\\ \alpha\beta\gamma=-1\end{cases}$, 5 marks

  or $\begin{cases}\alpha^3=5\alpha-1\\ \beta^3=5\beta-1\\ \gamma^3=5\gamma-1\\ \alpha^2+\beta^2+\gamma^2=10\end{cases}$, 5 marks

2. A 21×21 table contains 21 copies of each of the numbers 1, 2, 3, . . . , 20 and 21. The sum of all the numbers above the main diagonal (diagonal from the top-left cell to the bottom-right cell) is equal to three times the sum of all the numbers below the main diagonal. Find the sum of all the numbers on the main diagonal of the table. 【**Submitted by *Central Jury***】

【Solution】

There are 21 numbers on the main diagonal, $21\times10=210$ numbers above it and 210 numbers below it. The sum of the largest 210 numbers is

$$21\times(12+13+14+15+16+17+18+19+20+21)=21\times165,$$

while the sum of the smallest 210 numbers is

$$21\times(1+2+3+4+5+6+7+8+9+10)=21\times55.$$

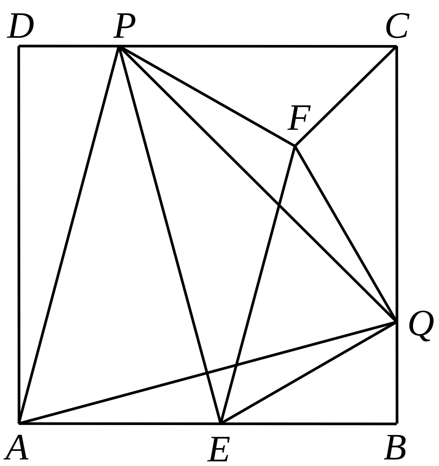
Since the former is exactly three times as large as the latter, the largest 210 numbers should all be located above the main diagonal and the smallest 210 numbers are all located below the main diagonal. In other words, every number on the main diagonal should be 11. Hence, the sum of the numbers on the main diagonal is $11\times21=231$.

Answer: 231

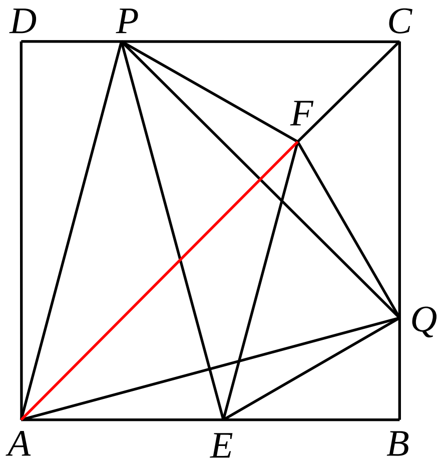
【Marking Scheme】

- Identified how many numbers are there above, below and on the main diagonal, 5 marks
- Calculated the largest possible sum of 210 numbers, 5 marks
- Calculated the smallest possible sum of 210 numbers, 5 marks
- Deduced that the numbers on the diagonal should all be 11 and arrived with the correct answer, or correct answer only (without any explanation), 5 marks.

3. In the figure below, *ABCD* is a square. Points *E*, *Q* and *P* are on sides *AB*, *BC* and *CD*, respectively, such that $PE\perp AQ$ and $\triangle AQP$ is equilateral triangle. Point *F* is inside $\triangle PQC$ such that $\triangle PFQ$ and $\triangle AEQ$ are congruent. If $EF=2$ cm, find the length of *FC*, in cm. 【**Submitted by *Bulgaria***】



【Solution】



From $AD = AB$ and $AP = AQ$, it follows $\triangle ADP \cong \triangle ABQ$ and $\angle DAP = \angle QAB = 15^\circ$. $\triangle AQP$ is an equilateral triangle, so PE is a symmetrical to AQ and $\triangle AQE$ is isosceles ($AE = EQ$). Therefore, $\angle QAE = \angle AQE = 15^\circ$ and $\angle AEQ = 150^\circ$.

Now, from $\triangle PFQ \cong \triangle AEQ$ it follows that F lies on the bisector of $\angle PCQ$, i.e. on diagonal AC.

$\angle FQE = \angle FQP + \angle PQE + \angle AQE = 15^\circ + 60^\circ + 15^\circ = 90^\circ$ and from $\triangle PFQ \cong \triangle AEQ$ we have $FQ = QE$, i.e. $\triangle FQE$ is isosceles right-angle triangle. Therefore $AE = FQ = EQ = \frac{EF}{\sqrt{2}} = \sqrt{2}$.

But $\triangle PCQ$ is isosceles right-angle triangle, so

$$\angle FQC = \angle PQC - \angle PQF = 45^\circ - 15^\circ = 30^\circ.$$

From $\angle AFE = 180^\circ - 45^\circ - (180^\circ - 45^\circ - 30^\circ) = 30^\circ$ and $\angle BAC = \angle ACB = 45^\circ$, it follows that triangles FAE and QCF are similar with coefficient $\frac{EF}{FQ} = \sqrt{2}$. Therefore,

$\sqrt{2} = \frac{EF}{FQ} = \frac{AE}{FC} = \frac{\sqrt{2}}{FC}$, i.e. $FC = 1$ cm.

Answer: 1 cm

【Marking Scheme】

- Correctly establish relationships of different triangles/angles (without any mistakes), up to 5 marks.
- Establish the relationship between *EF* and *FQ*, 5 marks
- Establish the fact that *FAE* and *FQC* are similar triangles, 5 marks
- Finding the correct answer with explanation, or 5 marks
- Correct answer only (without any explanation), 0 marks