Key Stage 3 - Team Contest

1. Let $f(x)=\dfrac{9^x}{9^x+3}$. Calculate $f\left(\dfrac{1}{2017}\right)+f\left(\dfrac{2}{2017}\right)+\ldots+f\left(\dfrac{2015}{2017}\right)+f\left(\dfrac{2016}{2017}\right)$.

【**Submitted by *Bulgaria***】

【Solution】

Notice that,

$$\begin{aligned} f(a)+f(1-a) &= \frac{9^a}{9^a+3}+\frac{9^{1-a}}{9^{1-a}+3} \\ &= \frac{9^a}{9^a+3}+\frac{9\times 9^{-a}}{9\times 9^{-a}+3} \\ &= \frac{9^a}{9^a+3}+\frac{3\times 9^{-a}}{3\times 9^{-a}+1} \\ &= \frac{9^a}{9^a+3}+\frac{3}{9^a+3}=1 \end{aligned}$$

So,

$$f\left(\frac{1}{2017}\right)+f\left(\frac{2016}{2017}\right)=f\left(\frac{1}{2017}\right)+f\left(1-\frac{1}{2017}\right)=1,$$

$$f\left(\frac{2}{2017}\right)+f\left(\frac{2015}{2017}\right)=f\left(\frac{2}{2017}\right)+f\left(1-\frac{2}{2017}\right)=1,$$

....,

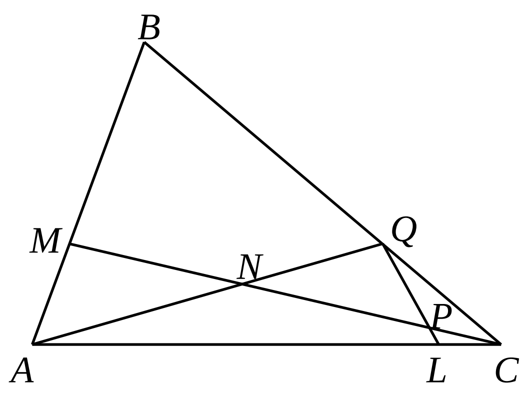
$$f\left(\frac{1008}{2017}\right)+f\left(\frac{1009}{2017}\right)=f\left(\frac{1008}{2017}\right)+f\left(1-\frac{1008}{2017}\right)=1$$

and $f\left(\dfrac{1}{2017}\right)+f\left(\dfrac{2}{2017}\right)+\cdots+f\left(\dfrac{2015}{2017}\right)+f\left(\dfrac{2016}{2017}\right)=1008.$

Answer: 1008

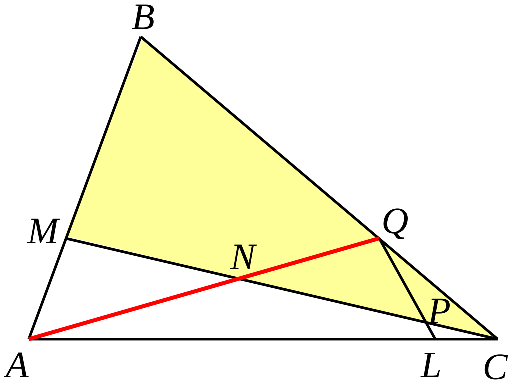
2. In $\triangle ABC$, point M is between A and B such that $AM : MB = 1 : 2$. Points N and P are between C and M such that $CN : NM = 3 : 2$, $CP : PM = 1 : 5$. Segments AN and BC intersect at point Q. Segments PQ and AC intersect at point L. Find the ratio $CL : LA$.

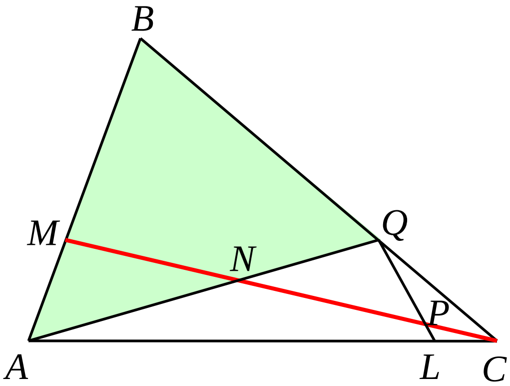
【**Submitted by *Bulgaria_SMG***】

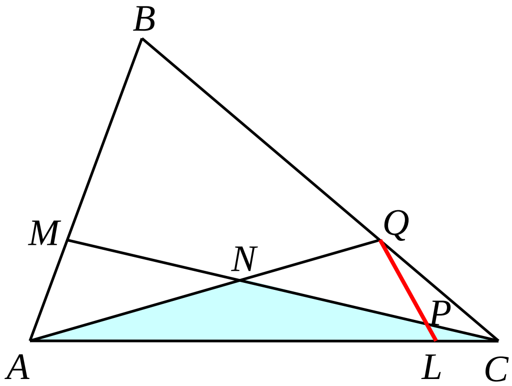


【Solution 1】

From the given, we know that $CP : PN : NM = 5 : 13 : 12$. We apply Menelaus theorem to $\triangle MBC$ and line AN, $\triangle ABQ$ and line CM, and to $\triangle ANC$ and the line PQ.







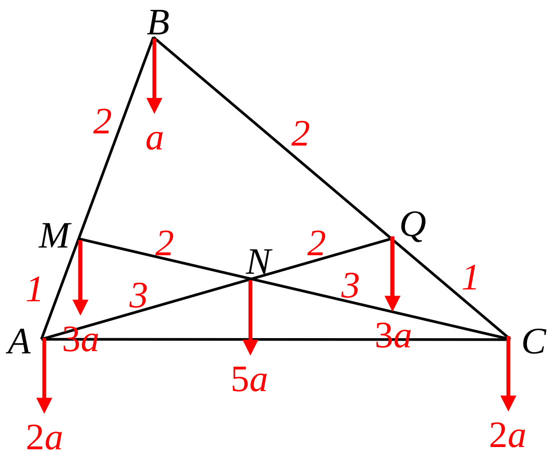
We have $\frac{BQ}{CQ}\times\frac{CN}{MN}\times\frac{AM}{AB}=1$, so $\frac{BQ}{CQ}=2$; $\frac{AM}{BM}\times\frac{BC}{QC}\times\frac{QN}{AN}=1$, and taking into account the previous equality, we have $\frac{QN}{AN}=\frac{2}{3}$. Finally, from $\frac{AQ}{NQ}\times\frac{NP}{CP}\times\frac{CL}{LA}=1$ and the previous equality, we obtain $\frac{CL}{LA}=\frac{2}{13}$.

【Marking Scheme】

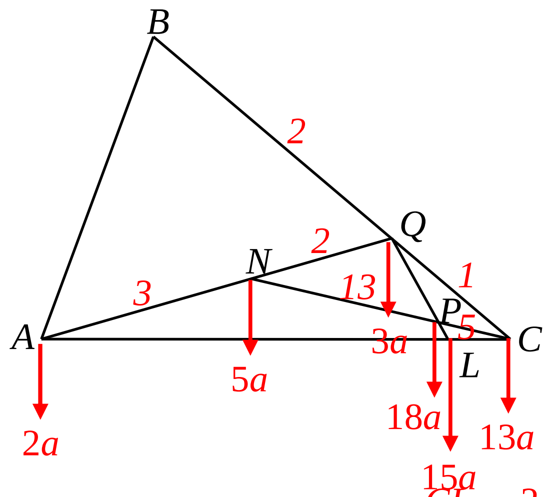
- Apply Menelaus theorem correctly, 10 marks
- Observe that $\frac{BQ}{CQ}=2$, 10 marks
- Observe that $\frac{QN}{AN}=\frac{2}{3}$, 10 marks
- Observe that $\frac{CL}{LA}=\frac{2}{13}$, 10 marks

【Solution 2】

From the given, we know that $CP : PN : NM = 5 : 13 : 12$. We can solve this by applying mass point geometry or "Law of the lever." Consider the system ABC and line AQ, CM. If the force component on point M is $3a$, then the force component on point A is $2a$ and the force component on point B is a. Hence, the force component on point C is $2a$ and the force component on point N is $5a$. So, the force component on point Q is $3a$. We get $\dfrac{AN}{NQ}=\dfrac{3}{2}$ and $\dfrac{BQ}{QC}=\dfrac{2}{1}$.



Next, consider the system AQC and line QL, CN. If the force component on point N is $5a$, then the force component on point C is $13a$ and the force component on point P is $18a$. Hence, the force component on point A is $2a$ and the force component on point Q is $3a$. So, the force component on point L is $15a$. We get $\dfrac{CL}{LA}=\dfrac{2}{13}$.



Answer: $\dfrac{CL}{LA}=\dfrac{2}{13}$

【Marking Scheme】

- Apply Mass Point Geometry or "Law of the lever" correctly, 10 marks
- Find the force components on each point of the system ABC and line AQ, CM, 10 marks
- Find the force components on each point of the system AQC and line QL, CN, 10 marks
- Observe that $\dfrac{CL}{LA}=\dfrac{2}{13}$, 10 marks

3. Find the largest integer p such that $14^{2017}+2^{2017}$ is divisible by 2^p.

【**Submitted by *Bulgaria_FPMG***】

【Solution】

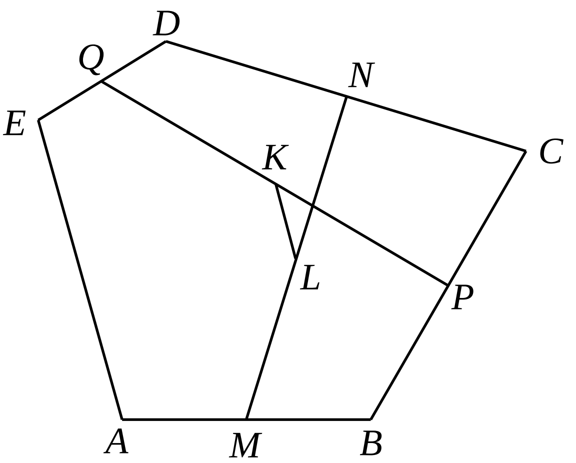
$14^{2017}+2^{2017}=2^{2017}\left(7^{2017}+1\right)$ and $7^{2017}\equiv 7\times\left(7^2\right)^{1008}\equiv 7(\text{mod}\,8)$, therefore,

$14^{2017}+2^{2017}\equiv 0\left(\text{mod}\,2^{2020}\right)$, but $7^{2017}+1\equiv 7\times\left(7^2\right)^{1008}\equiv 7+1\equiv 8\left(\text{mod}\,2^4\right)$.

Answer: 2020.

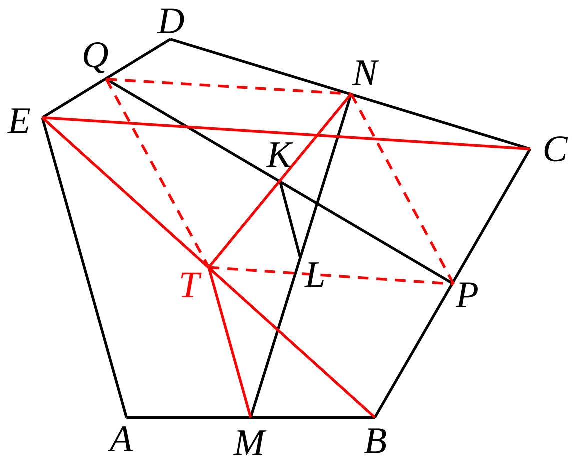
4. In pentagon *ABCDE*, points *M*, *P*, *N* and *Q* are midpoints of *AB*, *BC*, *CD* and *DE* respectively. While points *K* and *L* are midpoints of *QP* and *MN*, respectively, as shown in the figure below. If $KL = 25\text{cm}$, find the length of *EA*, in cm.

【Submitted by *Bulgaria*】



【Solution】

Connect *BE* and *CE*. Let *T* be the midpoint of *BE*. Connect *QN*, *NP*, *PT* and *TQ*.



In triangle *ABE*, $TM = \frac{1}{2}AE$ and $TM // AE$.

In triangle *ECD*, $QN = \frac{1}{2}CE$ and $QN // CE$.

In triangle *BEC*, $TP = \frac{1}{2}CE$ and $TP // CE$.

So, *TPNQ* is parallelogram.

Connect *NT* and *TM*.

In triangle *NTM*, $KL = \frac{1}{2}TM$ and $KL // TM$ since *K* is the midpoint of *QP* and *L* is the midpoint of diagonal *NT*. Hence, $KL = \frac{1}{4}AE$, i.e. $EA = 4KL = 100\text{cm}$.

Answer: 100 cm

【Marking Scheme】

- Plot point *T*, 10 marks
- Show that *TPNQ* is parallelogram, 10 marks
- Observe that $KL = \frac{1}{2}TM$, 10 marks
- Observe that $KL = \frac{1}{4}AE$, 5 marks
- Get the correct answer, 5 marks.

5. Let x and y be positive integers, where $0<x<y<2018$. How many ordered pairs (x, y) are there such that $x^2+2018^2=y^2+2017^2$? 【**Submitted by *Bulgaria_SMG***】

【Solution】

The given equation is transformed to $y^2-x^2=2018^2-2017^2$,

so that $(y+x)(y-x)=4035=3\times5\times269$.

Because $y+x$ and $y-x$ are positive integers, it follows

$$\begin{cases}y+x=5\times269\\ y-x=3\end{cases} \text{ or } \begin{cases}y+x=3\times269\\ y-x=5\end{cases} \text{ or } \begin{cases}y+x=269\\ y-x=15\end{cases} \text{ or } \begin{cases}y+x=3\times5\times269\\ y-x=1\end{cases}$$

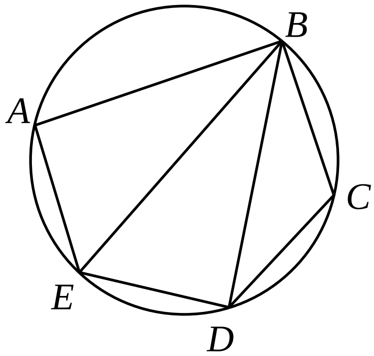
Solve them and get

$$\begin{cases}x=671\\ y=674\end{cases} \text{ or } \begin{cases}x=401\\ y=406\end{cases} \text{ or } \begin{cases}x=127\\ y=142\end{cases} \text{ or } \begin{cases}x=2017\\ y=2018\end{cases}$$

But the last pair (2017, 2018) does not satisfy the conditions, thus, there are only 3 pairs of positive integers (x, y).

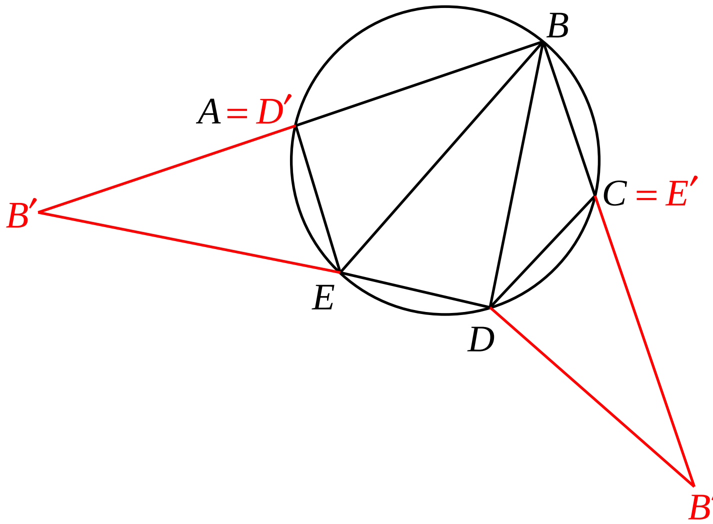
Answer: 3 pairs

6. Points A, B, C, D and E are on the circumference. Chord AC is a diameter of the circle, as shown in the figure below. If $\angle ABE=\angle EBD=\angle DBC$, $BE=16$ cm and $BD=12\sqrt{3}$ cm, find the area of pentagon $ABCDE$. 【**Submitted by *Indonesia***】

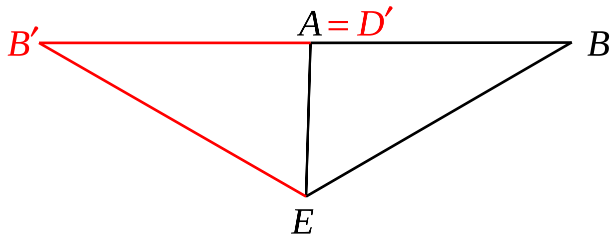


【Solution】

Since $\angle ABE=\angle EBD=\angle DBC$ and $\overset{\frown}{AC}$ is divided into three equal lengths, so, $AE=ED=DC$. $ABCD$ is a cyclic quadrilateral, so $\angle A+\angle D=180°$ and $\angle E+\angle B=180°$.



Look at triangle $\triangle BED$, since $AE = ED$, so if we rotate $\triangle BED$ clockwise, with center of rotation a point E, then point D placing point A and form a new triangle $\triangle BED'$ as shown in the figure below.

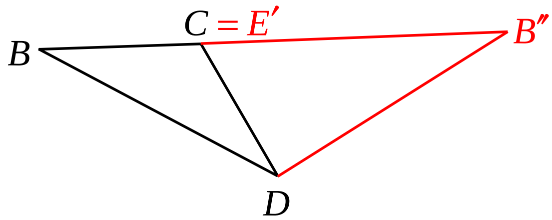


And since $BE = B'E$, and $\angle EBD' = \angle EB'D'$ so $\triangle BEB'$ is an isosceles triangle. $\angle ABE = \angle EBD = \angle DBC = 30°$,

and altitude of $\triangle BEB'$ is equal to $\frac{1}{2}BE = 8$ cm, $BB' = 16\sqrt{3}$ cm

The area of $\triangle BEB' = \dfrac{16\sqrt{3}\times 8}{2} = 64\sqrt{3}$ cm^2

If we do the same action for $\triangle BED$ counter clockwise with central of rotation at point D, so point E placing point C and form a new triangle $\triangle BDB''$ as shown in the figure below.



And since $BD = B''D$, and $\angle DB''E' = \angle DBE' = 30°$, $\triangle BDB''$ is an isosceles triangle. The altitude of $\triangle BDB''$ is equal to $\frac{1}{2}BD = 6\sqrt{3}$ cm, so $BB'' = 36$ cm. The area of $\triangle BDB'' = \dfrac{36\times 6\sqrt{3}}{2} = 108\sqrt{3}$ cm^2.

The area of $\triangle BED = \dfrac{BD\times BE\times \sin 30°}{2} = \dfrac{16\times 12\sqrt{3}\times \frac{1}{2}}{2} = 48\sqrt{3}$ cm^2.

Hence, the area of pentagon $ABCDE$ = Area $\triangle BEB'$ + Area $\triangle BDB''$ – area $\triangle BED$

$= 64\sqrt{3} + 108\sqrt{3} - 48\sqrt{3} = 124\sqrt{3}$ cm^2

Answer: $64\sqrt{3} + 108\sqrt{3} - 48\sqrt{3} = 124\sqrt{3}$ cm^2

【Marking Scheme】

- Found the area of $\triangle BEB'$, 10 marks
- Found the area of $\triangle BDB''$, 10 marks
- Found the area of $\triangle BED$, 10 marks
- Correct answer, 10 marks

7. A 10×10 chessboard is dissected into thirty-three 1×3 or 3×1 rectangles and one unit square. In how many different positions can this unit square be, if the chessboard may not be reflected or rotated? 【**Submitted by *Central Jury***】

【Solution】

We first label the 100 squares A, B or C diagonal by diagonal in two ways, as shown in the diagram below. In each labelling, there are 34 As, 33 Bs and 33 Cs. Since each rectangle covers one square with label A, one with B and one C, the 1×1 square must occupy a square with label A in both labellings. There are 16 possible positions for it. Each position is attainable. Just add three 1×3 rectangles in the same row, and fill the remaining three groups of three rows with 3×1 rectangles.

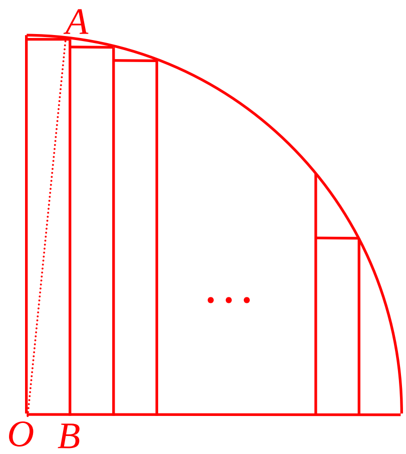
A	B	C	A	B	C	A	B	C	A
B	C	A	B	C	A	B	C	A	B
C	A	B	C	A	B	C	A	B	C
A	B	C	A	B	C	A	B	C	A
B	C	A	B	C	A	B	C	A	B
C	A	B	C	A	B	C	A	B	C
A	B	C	A	B	C	A	B	C	A
B	C	A	B	C	A	B	C	A	B
C	A	B	C	A	B	C	A	B	C
A	B	C	A	B	C	A	B	C	A

A	B	C	A	B	C	A	B	C	A
C	A	B	C	A	B	C	A	B	C
B	C	A	B	C	A	B	C	A	B
A	B	C	A	B	C	A	B	C	A
C	A	B	C	A	B	C	A	B	C
B	C	A	B	C	A	B	C	A	B
A	B	C	A	B	C	A	B	C	A
C	A	B	C	A	B	C	A	B	C
B	C	A	B	C	A	B	C	A	B
A	B	C	A	B	C	A	B	C	A

Answer: 16

8. Prove the inequality: $\sqrt{99\times101}+\sqrt{98\times102}+\ldots+\sqrt{1\times199}<\frac{100^2\pi}{4}$.【**Submitted by *Bulgaria_SMG***】

【Solution】



Consider a quarter circle with radius 1.

Now, inscribe a row-like figure consisting of 99 rectangles, each of them having a base length $\frac{1}{100}$.

The area of the first rectangle is:

$S_1=OB\times AB=OB\sqrt{1-OB^2}=\frac{\sqrt{99\times101}}{100^2}$,

the area of the second rectangle is:

$S_2=\frac{1}{100}\sqrt{1-(\frac{2}{100})^2}=\frac{\sqrt{98\times102}}{100^2}$

and so on. The area of the last one is $S_{99}=\frac{1}{100}\sqrt{1-(\frac{99}{100})^2}=\frac{\sqrt{1\times199}}{100^2}$.

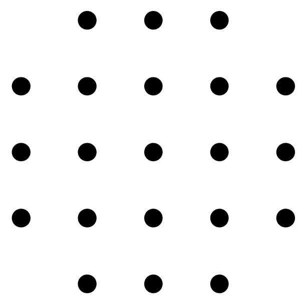
So, the total area of the figure is less than $\frac{1}{4}$ of the circle area, i.e.

$$\frac{\sqrt{99\times101}}{100^2}+\frac{\sqrt{98\times102}}{100^2}+\cdots+\frac{\sqrt{1\times199}}{100^2}<\frac{\pi}{4}.$$

【Marking Scheme】

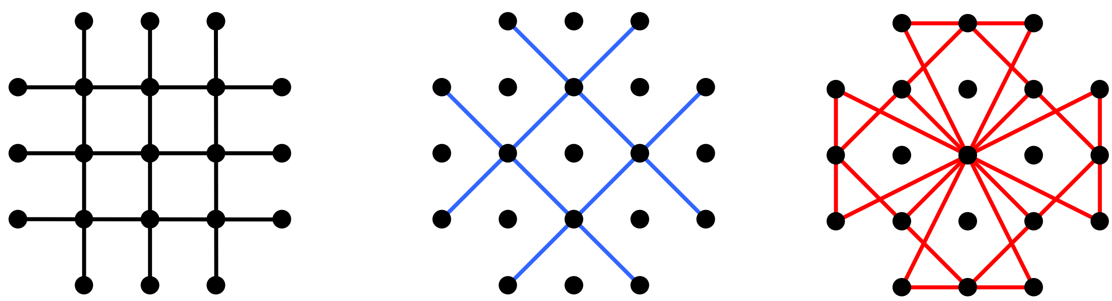
- Construct a quarter of a circle and inscribe the rectangles, 15 marks.
- Find the rules of the areas of the rectangles, 15 marks.
- Conclude the inequality, 10 marks.

9. A computer randomly chooses three different points on the given grid (all points have the same chance of being chosen). Let $\frac{p}{q}$ be the probability to form a triangle with these points (this fraction is written in its irreducible form). Find the sum of p and q. 【**Submitted by *Central Jury***】



【Solution】

There are $C_3^{21}=1330$ ways of choosing 3 points out of 21. They will form a triangle if not all three are on the same line. We can see that there are $C_3^5\times6=60$ ways to choose three points on one of the 6 lines (black) having exactly 5 points of the grid, $C_3^4\times4=16$ ways to choose three points on one of the 4 lines (blue) having exactly 4 points and $C_3^3\times14=14$ ways of choosing three points on one of the 14 lines (red) having exactly 3 points.



Thus, the number of ways in which a triangle can be formed is $1330-60-16-14=1240$. The probability is then $\frac{p}{q}=\frac{124}{133}$, and so $q+p=257$.

Answer: 257

10. Jane has 12 marbles, where in one is fake. She are not certain if the fake marble is heavier or lighter than the real marble. What is the minimum number of weightings needed to find the fake marble and determine whether the fake marble is heavier or lighter than the real marble? Explain your answer.

【Solution】

Jane would need to divide the 12 marbles into three groups (A_1,A_2,A_3,A_4), (B_1,B_2,B_3,B_4), and (C_1,C_2,C_3,C_4).

She begins by balancing A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4. If both balance, then she would know that one of C_1, C_2, C_3, C_4 is fake.
Now, she chooses 3 marbles from C_1, C_2, C_3, C_4 (assume that she has chosen C_1, C_2, C_3) and balance it against any 3 from of the known genuine marbles (assume that she chose A_1, A_2, A_3). If they balance, so the fake one is C_4. Otherwise, if they don't balance and that C_1, C_2, C_3 is heavier to A_1, A_2, A_3, she can conclude that the fake one is heavier and she chooses 2 from C_1, C_2, C_3 and do the weighing (assume she chose C_1 and C_2, if they balance, the fake marble is C_3, otherwise, the fake one is the heavier marble. (it's the same way scenario if one out of the set is lighter).

Now, suppose that she balances A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 and it didn't balance, so C_1, C_2, C_3, C_4 are all genuine. Suppose that A_1, A_2, A_3, A_4 was heavier than B_1, B_2, B_3, B_4. For her second balance, she replaces 3 marbles from A_1, A_2, A_3, A_4 (suppose she has chosen A_1, A_2, A_3) with 3 marbles from C_1, C_2, C_3, C_4 (suppose she has chosen C_1, C_2, C_3), and in addition, swap A_4 to one from B_1, B_2, B_3, B_4 (suppose she choose B_4). So, for the second balance we do weighing C_1, C_2, C_3, B_4 and B_1, B_2, B_3, A_4.
If it balances, she knows that one of the one from 3 balls A_1, A_2, A_3 is fake and heavier, then he only need to know a third weighing to know the fake one.
If C_1, C_2, C_3, B_4 is lighter, one of the 2 balls swapped (A_4 or B_4) is fake, so on the third weighing , weigh one marble from C_1, C_2, C_3, C_4 with A_4, if it balance, we know that B_4 is fake and lighter, if not, then A_4 is fake and heavier. If B_1, B_2, B_3, A_4 is still lighter, she would know that one of B_1, B_2, B_3 is lighter, then she only needs the third weighing to know the fake marble.
In any case, minimum three balances are required.

Answer: 3 weighings

【Marking Scheme】

- Reason why we cannot do with 2 weightings, 10 marks
- Give a solution with 3 weightings , up to 30 marks (partial credit to be given depending on progress)
- Show a solution with 4 weightings, which of course is not the right solution, 10 marks
- Solutions for 5 weightings or more, 0 marks.