Key Stage 2 - Individual Contest

1. A mathematics teacher originally designed a 90-minute exam which contains 75 questions. How many questions should be removed from the exam if the teacher wants to create a 60–minute exam with the same average time for each question as in the original 90–minute exam? 【**Submitted by *Bulgaria_FPMG***】

【Solution】

If the 75-question exam is to be taken in 90 minutes, then, the new exam is $\frac{60}{90}=\frac{2}{3}$ long of the original exam. So, there should be $75\times\frac{2}{3}=50$ questions in the new exam, which means 75 – 50 = 25 questions will be removed.

Answer:25

2. What is the remainder when $1+2+2^2+2^3+\ldots+2^{2017}$ is divided by 5?【Submitted by Blomfontein, South Africa】

【Solution】

Notice that $2^4\equiv 1\ (\text{mod } 5)$ and $1+2+2^2+2^3\equiv 0\ (\text{mod } 5)$.
Since there are 2018 terms in the series and $2018=4\times 504+2$, then,
$1+2+2^2+2^3+\ldots+2^{2017}\equiv 1+2\equiv 3\ (\text{mod } 5)$.

Answer: 3

2. Alex, Bert, Carlos and David are suspects of a bank robbery. It has been found out that:
   (a) If Bert is guilty, then Carlos is a partner in crime;
   (b) If Carlos is guilty, then either Alex is a partner in crime, or Bert is innocent;
   (c) If David is innocent, then Bert is guilty and Alex is innocent;
   (d) If David is guilty, then Bert is also guilty.

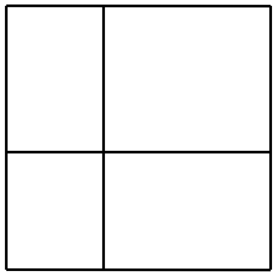
   Among the four suspects, how many are innocent?【**Submitted by *Central Jury***】

【Solution】

From conditions (c) and (d), we deduce that Bert is guilty. So, Carlos is also guilty from condition (a). Moreover, Alex is also guilty from condition (b). Now, in condition (c) again, David is also guilty since Bert and Alex are guilty. Thus, all of them are guilty, i.e. none of them are innocent.

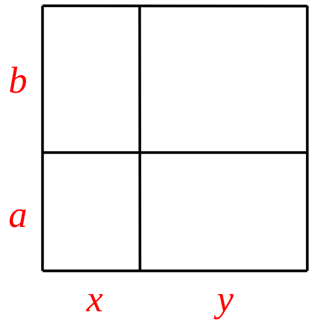
Answer:0

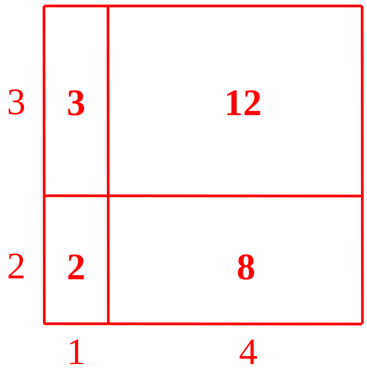
3. A square is divided by lines parallel to its sides into 4 rectangles as shown in the figure below. The length of a side of the square is known to be an integer, in cm., while the areas of the four rectangles are pairwise distinct positive integers. Find the smallest possible value of the area of the original square, in cm^2. 【**Submitted by *Central Jury***】



【Solution 1】

Denote the segments as shown in the figure. Observe that the values of *a*, *b*, *x* and *y* must be pairwise distinct positive integers, otherwise, we will get at least two rectangles with the same area. Since we want to find the smallest possible value of the area of the square, we may take the values of *a*, *b*, *x* and *y* as 1, 2, 3 and 4. Since $a+b=x+y$ and $1+4=2+3$, we can take $a=2$, $b=3$, $x=1$ and $y=4$ as the following figure:





So, the smallest possible value of the area of the square is $5\times5=25\text{cm}^2$.

【Solution 2】

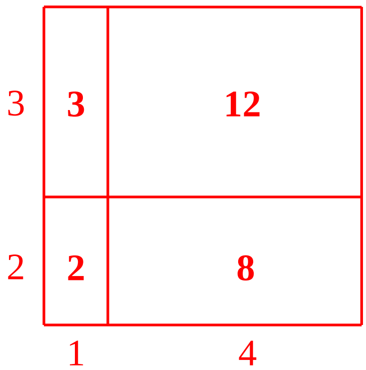
Let the length of a side of the square be *a* cm. It is obvious that $a\neq1$.

If $a=2=1+1$, then the areas of all four rectangles are 1 cm^2, which is a contradiction.

If $a=3=1+2$, then the areas of two of the rectangles is 2 cm^2, which is a contradiction.

If $a=4=1+3=2+2$, then there are at least two rectangles that will have the same area, which is a contradiction.

If $a=5=1+4=2+3$, then we can get a solution as the following figure:



So, the smallest possible value of the area of the square is $5\times5=25\text{cm}^2$.

Answer: 25 cm^2

4. Find the greatest positive integer made up of different digits, such that if we remove the leftmost digit, its value would become $\frac{1}{5}$ of the original number.

【Submitted by *Mongolia*】

【Solution】

Suppose the leftmost digit is a and remaining number is A, then we have $\overline{aA}=5A$. Subtracting A from both sides of the equation, we get $\overline{a\underbrace{0...0}_{n}}=4A$, or $25\times\overline{a\underbrace{0...0}_{n-2}}=A$. If $n>3$, then the number A will have the same last two digits (which is 0), so it's not possible. If $n=3$, then the number will be $A=250a$ should have at least 3 different digits. From here, we can easily get the maximum which occurs when $a=3$.

Answer: 3750

5. Three brothers have some candies in a pile. On the first day, the eldest brother divided all the candies into three equal parts with a remainder of one candy, and then this eldest brother took one part. On the second day, the youngest brother divided all the remaining candies into five equal parts with one remaining candy, and then this youngest brother took two parts. On the third day, the third brother divided the remaining candies into five equal parts with no extra candy left. After the third brother had taken three parts of candies, what is the minimum number of candies left in all? **【Submitted by *Central Jury*】**

【Solution】

From the way the eldest and the youngest brother divided the pile, we can let $15n+1$ be the number of candies in the original pile. After the first day, there would be $10n+1$ candies left. After the second day, there would be $6n+1$ candies left. Now, we want this to be divisible by 5, so the smallest possible n that satisfies this condition is when $n=4$. Thus, after the second day, there will be $6n+1=25$ candies left. After the third brother had taken 3 parts, the minimum number of candies left is $25\times(1-\frac{3}{5})=10$ candies.

Answer: 10

6. How many numbers from all 2-digit, 3-digit and 4-digit numbers are neither divisible by 12 nor by 30? **【Submitted by *Bulgaria_FPMG*】**

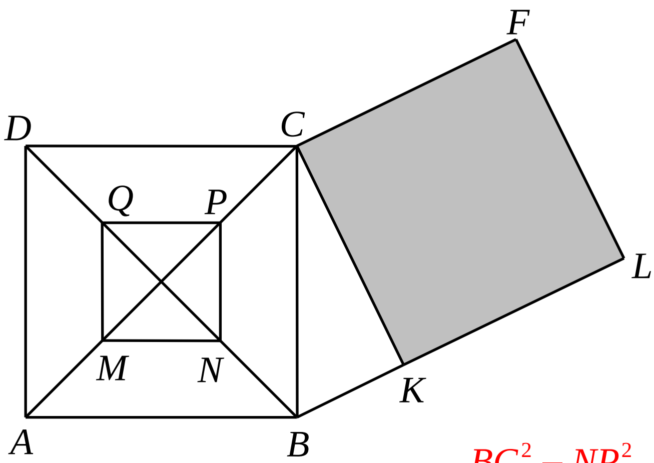
【Solution】

There is a total of 9990 numbers with 2, 3 or 4 digits. Among them, there are 833 numbers that are divisible by 12, 333 numbers that are divisible by 30 and 166 numbers that are divisible by $60=[12,30]$.

So, there are $9990-833-333+166=8990$ numbers.

Answer: 8990

7. If $ABCD$ and $KLFC$ are two squares so that B, K and L are collinear. Points M and P are on AC, points N and Q are on BD so that $MNPQ$ is also a square, as shown in the figure below. If $MN = BK$ and area of quadrilateral $BCPN$ is 2017 cm^2, find the area of square $KLFC$, in cm^2. **【Submitted by *Bulgaria_FPMG*】**



【Solution】

Observe that the area of quadrilateral $BCPN$ is equal to $\dfrac{BC^2 - NP^2}{4}$ and the area of square $KLFC$ is equal to KC^2. Since B, K and L are collinear, $\angle BKC = 90°$. Thus, by the Pythagorean Theorem,

$$KC^2 = BC^2 - BK^2 = BC^2 - MN^2 = BC^2 - NP^2 = 4\times 2017 = 8068 \text{ cm}^2.$$

Answer: 8068 cm^2

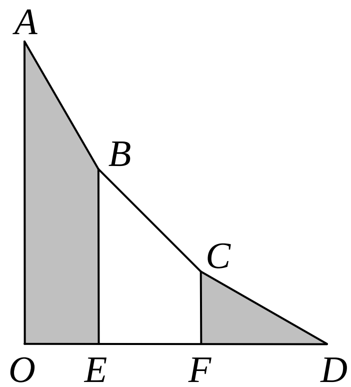
8. There are 70 boys and 30 girls that participated in a school chess tournament. In each round, all participants are grouped into 50 pairs and each pair plays one game. After three rounds, 21 games were played between girls. How many games were played between boys? **【Submitted by *Mongolia*】**

【Solution】

After 3 rounds, 30 girls played $30\times 3 = 90$ games altogether. Here, 21 games played between girls are counted twice. The remaining $90 - 2\times 21 = 48$ games are played between boys and girls. Now consider $70\times 3 = 210$ games played by 70 boys. As we know, 48 games were played between boys and girls. So, the remaining $210 - 48 = 162$ games were played between boys. By the same reasoning earlier, we should divide this number by 2, so between boys, they play 81 games.

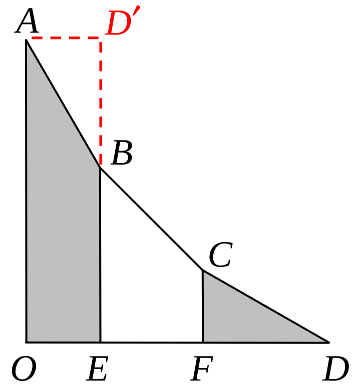
Answer: 81

9. If OA and OD are perpendicular segments each with length of 20 cm. AB, BC and CD are segments of equal length such that $\angle OAB = 30° = \angle ODC$. E and F are points on OD such that BE and CF are perpendicular to OD. Let $[P]$ denote the area of the polygon P. Determine $\dfrac{[OABE]+[CDF]}{BC}$, in cm.



【Submitted by *Central Jury*】

【Solution】



Notice by symmetry that BC makes an angle of 45° with OD. Observe that we can reflect and rotate triangle CFD and move it to ABD' so that CD coincides with AB to form a rectangle $AOED'$. Now let $AB = BC = CD = 2a$ cm. Then $OE = a$ cm and hence the area of rectangle $AOED'$ is $20a$ cm^2. Thus, $\frac{[OABE]+[CDF]}{BC} = \frac{20a}{2a} = 10$ cm.

Answer: 10 cm

10. In today's school fair, donuts were given out for snacks. Each student got 2 donuts, while each adult got 3 donuts. All of the girl students ate just one of their donuts, and gave away their second one to the boy students. The adults ate all of their share of donuts. Finally, there are 23 boy students that ate 4 donuts and all of the remaining boy students ate 3 donuts. We know that a total of 237 people joined the fair and ate a total of 508 donuts. How many boy students attended the school fair? **【Submitted by *Bulgaria_FPMG*】**

【Solution】

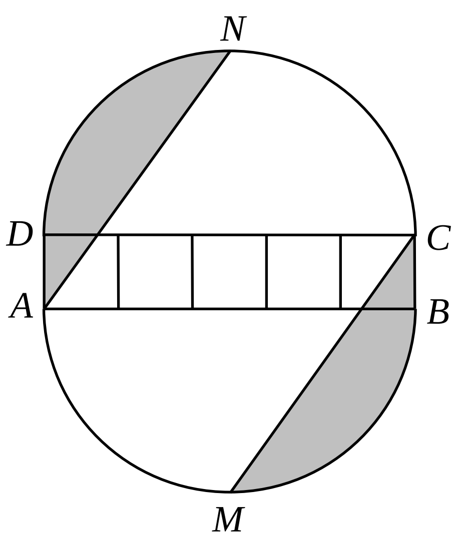
Let a be the number of adults, b be the number of boys and c be the number of girls. Then,

$2(b+c)+3a=508$, $a+b+c=237$, $2b+c=23\times4+(b-23)\times3$.

Hence, $b+c=3\times237-508=203$ and $c=23+b$. So, $b=90$.

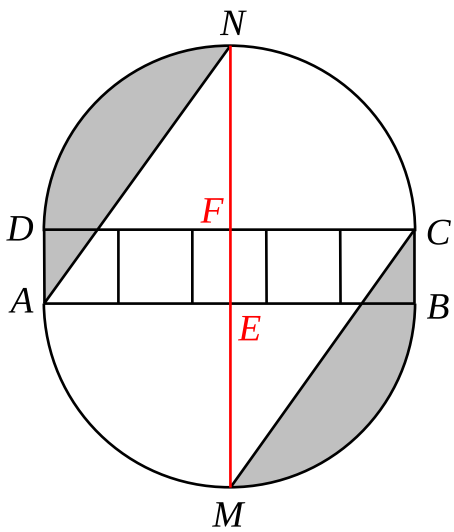
Answer: 90

11. In the figure below, $AB = 5AD = 70$ cm. DNC and AMB are two semicircles and points M and N are the midpoints of arcs DC and AB, respectively. Find the area of the shaded region, in cm^2. Take $\pi = \frac{22}{7}$. **【Submitted by *Central Jury*】**



【Solution】

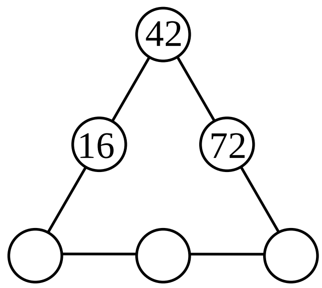
Observe that $AD = 14$ cm and the radius of the two semicircles are both 35 cm. So, the area of the figure is $70\times 14+\pi\times 35^2 = 980+1225\pi$ cm². Connect *MN*. Let point *E* be the intersection of *AB* and *MN* and point *F* be the intersection of *CD* and *MN*. Therefore, triangle *AEN* and *CFM* are congruent right triangles and *NFC* and *MEA* are quarter circles. Since $AE = FC = 35$ cm and $NE = MF = 35+14 = 49$ cm, the area of the unshaded region is $\frac{1}{2}\times 35\times 49\times 2+\frac{1}{4}\times\pi\times 35^2\times 2 = 1715+\frac{1225}{2}\pi$ cm². Hence, the area of the shaded region is



$$(980+1225\pi)-(1715+\frac{1225}{2}\pi) = \frac{1225}{2}\pi - 735 = 1925-735 = 1190 \text{ cm}^2.$$

Answer: 1190 cm²

12. The diagram shows six circles lying in 3 different lines. Three of the circles are numbered 42, 16 and 72 while the remaining three circles should contain positive integers, such that product of the numbers written in each of the three lines are the same. How many different ways can this be done? 【**Submitted by *Mongolia***】



【Solution】

Denote the numbers in the remaining three circles from left to right to be *a*, *b* and *c*. Since $42\times 16\times a = 42\times 72\times c$, we have $2a = 9c$. This equality holds when $a = 9d$, $c = 2d$ for some positive integer *d*. Now, we should have $42\times 16\times 9d = 9d\times b\times 2d$ or $b\times d = 336$. Since $336 = 2^4\times 3\times 7$, there are $(4+1)(1+1)(1+1) = 20$ solutions of the equation $b\times d = 336$.

Answer: 20

13. Suppose one of the symbols ○, △, ▽ is inserted into each of the unit squares of the 2×6 chessboard. How many different ways can these symbols be inserted into the unit squares of the chessboard such that the same symbols are not placed between any two unit squares that share a common side? (Note: Rotation or flipping should be considered different.) 【**Submitted by *Central Jury***】

【Solution】

Let us determine how the symbols should be put into each column starting with the left-most column. Since the same symbol cannot be placed adjacent to each other, there are $3\times 2 = 6$ ways of filling the left-most column. For each *k* with $1\le k\le 5$ suppose the *k*-th column from the left is filled as:

$$\begin{vmatrix} A \\ B \end{vmatrix},$$

then the (*k*+1)-th column from the left must be filled by one of the following three ways:

$$\begin{vmatrix} B \\ A \end{vmatrix}, \begin{vmatrix} B \\ C \end{vmatrix}, \begin{vmatrix} C \\ A \end{vmatrix}.$$

Therefore, the number of ways of filling the boxes of the grid by the symbol ○, △ , ▽ is $6\times 3^5 = 3^6 \times 2 = 1458$.

Answer: 1458

14. How many ways can we choose 3 different numbers from the set {1, 2, 3, …, 24, 25} so that they have no common divisors greater than 1. 【**Submitted by *Mongolia***】

【Solution】

There is a total of $\binom{25}{3} = 2300$ different ways to choose 3 numbers $a < b < c$.

From the total, we should remove those numbers having common divisor greater than 1. Now, we work on removing common divisors that are prime numbers. If a prime number *p* divides *a*, *b*, *c*, then $c \ge 3p$, so we have $p \le \frac{25}{3}$ or $p = 2$, 3, 5, 7. Denote by N_p is set ordered triples $a < b < c$ all divisible by *p*. We have

$$|N_2| = \binom{12}{3} = 220, \ |N_3| = \binom{8}{3} = 56, \ |N_5| = \binom{5}{3} = 10, \ |N_7| = \binom{3}{3} = 1.$$

All these sets N_2, N_3, N_5, N_7 are disjoint, except the sets N_2, N_3 intersecting by the set N_6 having $\binom{4}{3} = 4$ triples.

So, $2300 - (220 + 56 + 10 + 1 - 4) = 2300 - 283 = 2017$ triples have no common divisors greater than 1.

Answer: 2017