Key Stage 2 - Team Contest

1. Let each of the letters *D, A, V, O, M, T, H* and *S* represent a distinct digit from 0 to 9 so that $\overline{DAVAO}$ and $\overline{MATHS}$ are 5-digit numbers and it satisfies:

$$\begin{array}{cccccc} & D & A & V & A & O \\ + & D & A & V & A & O \\ \hline & M & A & T & H & S \end{array}$$

Find the sum of all possible values for *T*.【**Submitted by *Bulgaria_FPMG***】

【Solution】

From the thousands digits, it is clear that *A* is 0 or 9.

If $A=0$, then $H=1$ from the tens digits and hence we have:

(i) $O+O=\overline{HS}>10$, i.e. $O\geq 6$ and $S=2$, 4, 6 or 8.

(ii) $V+V=T<10$, i.e. $2\leq V\leq 4$ and $T=4$, 6 or 8.

(iii) $D+D=M<10$, i.e. $2\leq D\leq 4$ and $M=4$, 6 or 8.

- If $D=2$, then $M=4$ and hence $V=3$. So $T=6$. Thus, $S=8$ and $O=9$. We get a solution:

$$\begin{array}{cccccc} & 2 & 0 & 3 & 0 & 9 \\ + & 2 & 0 & 3 & 0 & 9 \\ \hline & 4 & 0 & 6 & 1 & 8 \end{array}$$

- If $D=3$, then $M=6$.
  As $V=2$, then $T=4$ and hence, $S=8$. So $O=9$. We get a solution:

$$\begin{array}{cccccc} & 3 & 0 & 2 & 0 & 9 \\ + & 3 & 0 & 2 & 0 & 9 \\ \hline & 6 & 0 & 4 & 1 & 8 \end{array}$$

  As $V=4$, then $T=8$ and hence, $S=2$. So $O=6$, which is impossible.
- If $D=4$, then $M=8$ and hence, $T=6$. So $V=3$. Thus, $S=2$ and we can conclude that $O=1$, which is impossible.

If $A=9$, then $H=8$ and $O+O=S<10$. Since $H=8$, we have $S=2$, 4 or 6 and $1\leq O\leq 3$. We also have:

(i) $V+V+1=\overline{1T}\geq 10$, i.e. $5\leq V\leq 7$ and $T=1$, 3 or 5.

(ii) $D+D+1=M<10$. Since $A=9$, we have $1\le D\le 3$ and $M=3$, 5 or 7.

- If $V=5$, then $T=1$.
  As $M=3$, then $D=1$, which is impossible.
  As $M=7$, then $D=3$ and hence, $O=2$. So $S=4$. We get a solution:

$$\begin{array}{r} 3\ 9\ 5\ 9\ 2 \\ +\ 3\ 9\ 5\ 9\ 2 \\ \hline 7\ 9\ 1\ 8\ 4 \end{array}$$

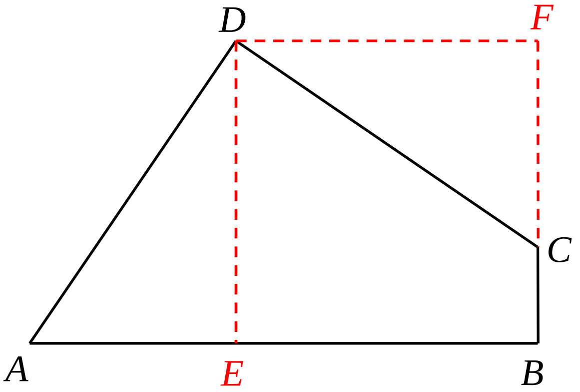
- If $V=6$, then $T=3$.
  As $O=1$, then $S=2$ and hence, $D=0$, which is impossible.
  As $O=2$, then $S=4$ and hence, $D=1$. So $M=3$, which is impossible.
- If $V=7$, then $T=5$ and hence, $M=3$. So $D=1$ and $O=2$, $S=4$. We get a solution:

$$\begin{array}{r} 1\ 9\ 7\ 9\ 2 \\ +\ 1\ 9\ 7\ 9\ 2 \\ \hline 3\ 9\ 5\ 8\ 4 \end{array}$$

Thus, there are four possible values for *T*, namely 6, 4, 1 and 5. So the answer is 16.

Answer: 16

2. The figure below shows a quadrilateral *ABCD* so that $\angle ADC=\angle ABC=90°$ and $AD=DC$. If the area of the quadrilateral *ABCD* is 196 cm^2, find the distance from *D* to *AB*, in cm. 【**Submitted by *Central Jury***】



【***Solution 1***】

Let point *E* on *AB* and point *F* on the extension of *BC* so that *DE*//*BC* and *DF*//*AB*. (10 Marks) Since $\angle ABC=90°$, $\angle AED=\angle DFC=90°$. Observe that $\angle ADE+\angle EDC=90°=\angle CDF+\angle EDC$, so $\angle ADE=\angle CDF$. Since $AD=DC$, triangle *ADE* and *CDF* are congruent (10 Marks) and hence $DE=DF$ (10 Marks). Thus, *DEBF* is a square and the area of *DEBF* is equal to the area of *ABCD*. Thus $DE=14$ cm since $DE^2=196=14^2$. (10 Marks)

【*Solution 2*】

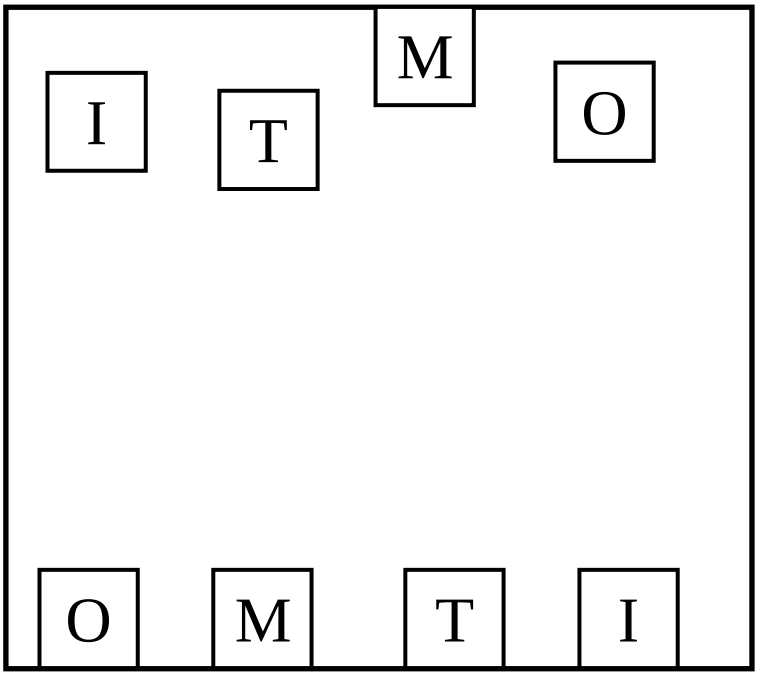
Let point E on AB so that $DE \perp AB$. Since $\angle ADC = \angle ABC = 90°$, $\angle DAE + \angle BCD = 360° - 90° - 90° = 180°$. Thus as we rotate triangle DAE 90° counterclockwise to triangle DCF so that DA and DC coincides (10 Marks) , then F, C and B are collinear. (10 Marks) So $BEDF$ is a square and the area of $BEDF$ is equal to the area of quadrilateral $ABCD$, $DE^2 = 196 = 14^2$. Hence $DE = 14$ cm. (10 Marks)

Answer: 14 cm

3. Connect each small box on the top with its same letter on the bottom with paths that do not cross one another, nor leave the boundaries of the large box.

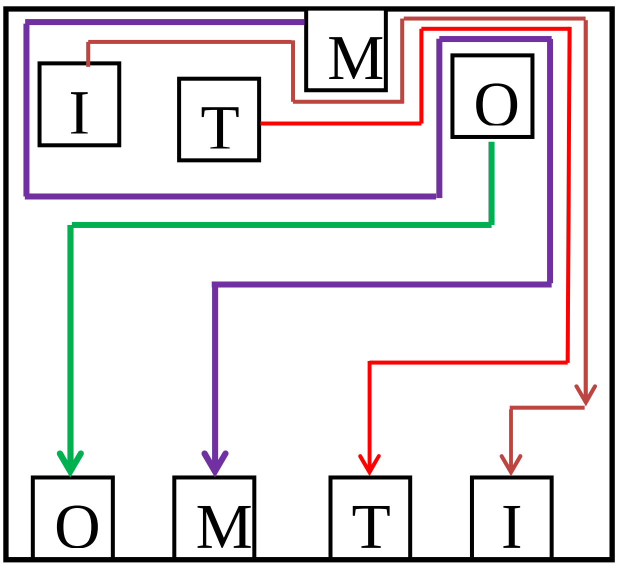
Connect each letter in the box on the top with the same letter on the bottom with paths not crossing over one another, nor paths not going outside the border .

【**Submitted by *Sri Lanka***】



【Solution】

One of many possible ways is shown in the diagram below.



4. Fill in each $\square$ with digit from the set {1, 2, 3, 4, 5, 6, 7, 8, 9} with no repetition into the following math operation:

$$\frac{1}{\square}+\frac{1}{\square+\frac{1}{\square}}+\frac{1}{\square+\frac{1}{\square+\frac{1}{\square}}}=\frac{M}{N}$$, where M and N are relatively prime.

Find greatest possible value of $M-N$. 【**Submitted by *Indonesia***】

【Solution】

Simplifying, we get $\frac{1}{a}+\frac{1}{b+\frac{1}{c}}+\frac{1}{d+\frac{1}{e+\frac{1}{f}}}=\frac{1}{a}+\frac{c}{bc+1}+\frac{ef+1}{def+d+f}$

From the above equation, we know $a=1$. To make $\frac{c}{bc+1}$ greatest we choose $c=9$ and $b=2$. To make $\frac{ef+1}{def+d+f}$ greatest, we choose $d=3$, $e=7$ and $f=8$. So, the equation becomes

$$\begin{aligned}\frac{1}{a}+\frac{c}{bc+1}+\frac{ef+1}{def+d+f}&=\frac{1}{1}+\frac{9}{2\times9+1}+\frac{7\times8+1}{3\times7\times8+3+8}\\&=1+\frac{9}{19}+\frac{57}{179}\\&=\frac{3401+1611+1083}{3401}\\&=\frac{6095}{3401}\end{aligned}$$

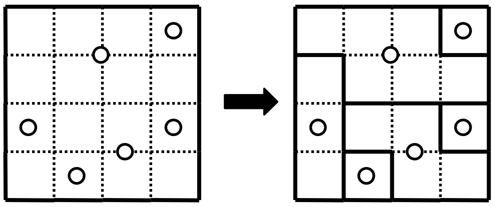
So, the value $M-N = 6095 - 3401 = 2694$

【Marking Scheme】

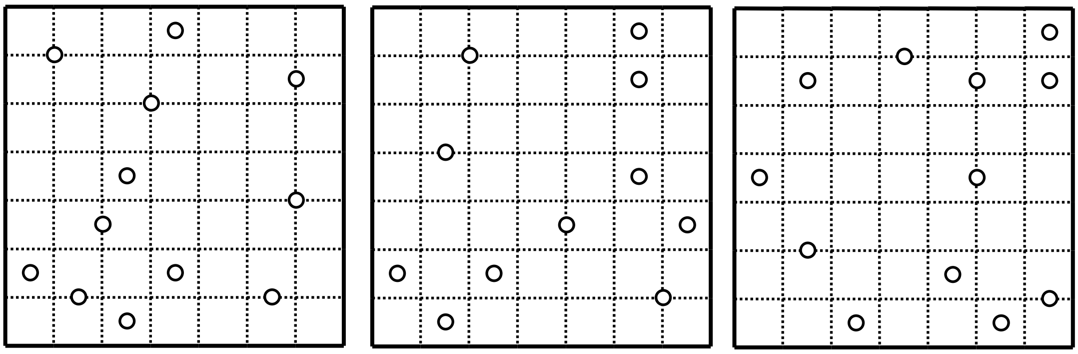
- Be able to give the correct value of a, 5 marks
- Be able to give the correct value of b and c, 5 marks
- Be able to give the correct value of d, e and f, 10 marks
- Be able to give the correct value of M and N, 10 marks
- Be able to give the correct difference $M-N$, 10 marks

5. Form edges along the dotted lines to create shapes so that each circle is the symmetry center (each shape when rotated 180 degrees along the circle, the shape appears identical) of the enclosed area. An example is shown below. Complete the three challenges below. 【**Submitted by *Indonesia***】

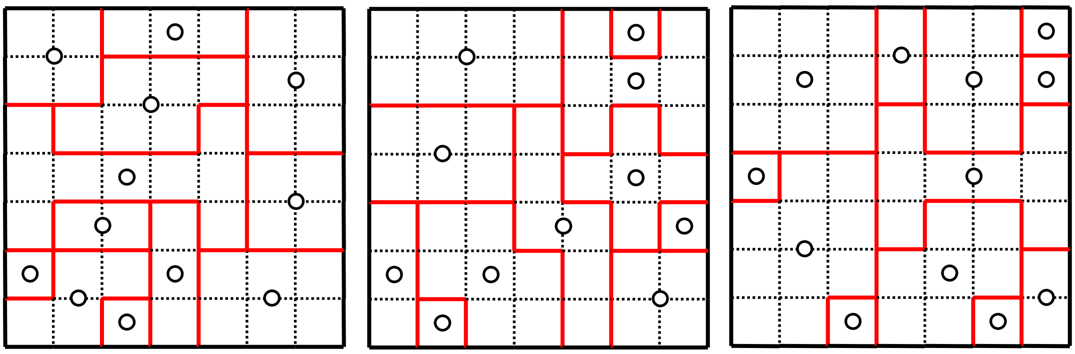
Example :



Challenge



【Solution】



【Marking Scheme】

- Each correct figure, 10 marks. 10 marks bonus for all 3 correct figures.

6. There are three 3-digit numbers $\overline{ABC}$, $\overline{BCD}$ and $\overline{CDE}$, where each different letter represents a different digit, so that $\overline{ABC}+\overline{BCD}+\overline{CDE}=2017$. Find the difference between the largest and smallest possible value of $\overline{ABCDE}$.
【**Submitted by *Mongolia***】

【Solution】

Observe that all of A, B and C are not equal to 0. Now we have

$$\begin{array}{r} A\ B\ C \\ B\ C\ D \\ +\quad C\ D\ E \\ \hline 2\ 0\ 1\ 7 \end{array}$$

Since sum of three different digits are not greater than $9+8+7=24$ and not less than $0+1+2=3$, there are following 4 cases:

(i) $\begin{cases} C+D+E=7 \\ B+C+D=11; \\ A+B+C=19 \end{cases}$ (ii) $\begin{cases} C+D+E=7 \\ B+C+D=21; \\ A+B+C=18 \end{cases}$

(iii) $\begin{cases} C+D+E=17 \\ B+C+D=10; \\ A+B+C=19 \end{cases}$ (iv) $\begin{cases} C+D+E=17 \\ B+C+D=20 \\ A+B+C=18 \end{cases}$

In case (i), $B=E+4$ and $A=D+8\ge 8$.

In case (ii), $B=E+14$, this is in contradiction with $B<10$.

In case (iii), $E=B+7$ and $A=D+9$. So $A=9$, $D=0$ and hence $C+E=17$, this is in contradiction with $C+E\le 7+8=15$.

In case (iv), $B=E+3$ and $D=A+2$. So $A\le 7$.

Thus only case (i) and (iv) can work.

To minimize $\overline{ABCDE}$, we consider case (iv) first. If $A=1$, then $D=3$ and we can get $B+C=17$ and $C+E=14$. Thus we can take $B=8$, $C=9$ and $E=5$. So the smallest possible value of $\overline{ABCDE}$ is 18935.

To maximize $\overline{ABCDE}$, we consider case (i) first. If $A=9$, then $D=1$ and hence $B+C=10$ and $C+E=6$. Thus we can take $B=8$, $C=2$ and $E=4$. So the largest possible value of $\overline{ABCDE}$ is 98214.

So the difference between the largest and the smallest possible value of $\overline{ABCDE}$ is $98214-18935=79279$.

Answer: 79279

【Marking Scheme】

- Find the largest possible value of $\overline{ABCDE}$, 15 marks
- Find the smallest possible value of $\overline{ABCDE}$, 15 marks
- Get the correct answer, 10 marks.

7. Andy wants to put some tokens (can put 0 token) on each of the unit squares of a 3×3 board, so that in any 2×2 subboard, the sum of the number of tokens is a prime number and all of those prime numbers are different. What is the least possible number of tokens that Andy should put on the board? Show one example. 【**Submitted by *Central Jury***】

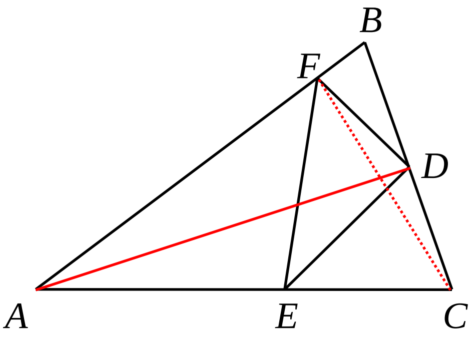
【Solution】

Let $2 \le p_1 < p_2 < p_3 < p_4 \le 7$ be the number of tokens in the 4 squares of 2×2 subboards. There is a 2×2 subboard with at least 7 tokens, therefore, Andy puts at least seven tokens. These is a possible way to put the tokens:

0	0	0
0	2	1
0	3	1

Answer: 7

8. The figure shows a triangle *ABC*. *D* is the midpoint of *BC* and *E* lies on *AC* such that *AE* : *EC* = 3 : 2. If *F* is a point of *AB* such that the area of triangle *DEF* is three times the area of triangle *BDF*, find the ratio of *AF* : *FB*. 【**Submitted by *Central Jury***】



【***Solution***】

Connect *CF* and *AD* and let the area of triangle *ABC* be 10. Then the area of triangle *ADC* is 5 and hence the area of triangle *CDE* is 2. Thus the area of quadrilateral *ABDE* is 8.

Suppose *AF* : *FB* = *a* : *b*. Then the area of triangle *AFC* is $\frac{10a}{a+b}$ and hence the area of triangle *AEF* is $\frac{6a}{a+b}$.And the area of triangle *BFC* is $\frac{10b}{a+b}$ and hence the area of triangle *BDF* is $\frac{5b}{a+b}$. Thus the area of triangle *DEF* is $\frac{15b}{a+b}$ and hence the area of quadrilateral *BDEF* is $\frac{20b}{a+b}$. Since quadrilateral *ABDE* is formed by triangle *AEF* and quadrilateral *BDEF*, we have $\frac{20b}{a+b}+\frac{6a}{a+b}=8$, i.e. $20b+6a=8a+8b$, or $a=6b$. So *AF* : *FB* = *a* : *b* = 6 ∶ 1.

Answer: 6 ∶ 1

【Marking Scheme】

- This problem has many different potential solutions and approaches
- Expressing the area of triangle *DEF* in terms of *AF*:*FB* → up to 10 Marks (depending of progress)
- Expressing the area of triangle *BDF* in terms of *AF*:*FB* → up to 10 Marks (depending on progress)
- Solving for *AF*:*FB* based on the 2 previous relationships → up to 10 Marks (depending on progress)
- Guessing the ration → 10 Marks

9. Use each of the digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 once to form a 2-digit number, a 3-digit number and a 4-digit number, respectively, so that when the first two numbers are multiplied together, the result would be the third number. For example, $12\times483=5796$.
Aside from the example given above, list down all possible combinations.
【**Submitted by *Thailand***】

【Solution】

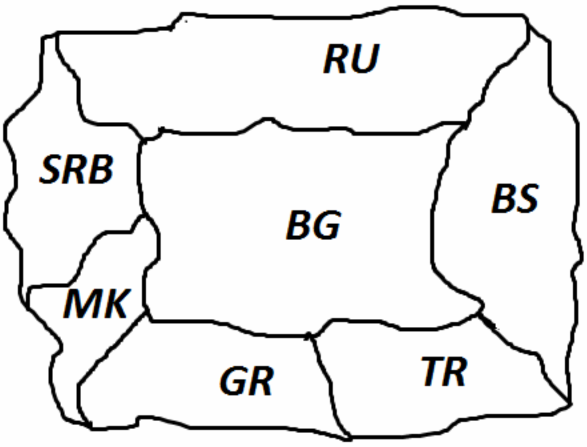
There are six more possible combinations (as shown below), aside from the example given in the problem.

$$18\times297=5346$$
$$27\times198=5346$$
$$28\times157=4396$$
$$39\times186=7254$$
$$48\times159=7632$$
$$42\times138=5796$$

【Marking Scheme】

- Each correct combination, 6 marks. 4 marks bonus for all 6 correct figures.

10. The picture shows a part of the Balkan peninsula. In how many ways can we color each region with one of 4 colors, so that every 2 neighboring regions are colored with different colors? 【**Submitted by *Bulgaria_FPMG***】



【Solution 1】

WLOG let BG be colored in color *a*, RU in color *b* and SRB in color *c* which we can do in $4\times3\times2=24$ ways. (10 Marks) Depending on the ways we color MK, we can color GR, TR, BS as following, there are 11 ways. (20 Marks)

MK	GR	TR	BS	
b	c	b	c	1
			d	2
		d	c	3
	d	b	c	4
			d	5
		c	d	6
d	b	c	d	7
		d	c	8
	c	b	c	9
			d	10
		d	c	11

Finally, the total number is: $24(3\times 2+5\times 1)=264$ ways.

【Solution 2】

Let's use an approach where we fix the color for a specific country and find out how many choices we have for the adjacent countries. We will use the order RU →SRB→BG→MK→GR→TR→BS

There are 4 colors available for RU. Choose one of them (say A).
There are 3 colors now available for SRB. Choose B.
There are 2 colors available now for BG (as A and B are taken by adjacent countries). Say we choose C.
So far we have 4 x 3 x 2 = 24 combinations for (A,B,C)
For MK, GR and TR we have 2 options for coloring each once we have fixed the previous country, namely:

MK→ two colors as it cannot be B and C

GR → there is a choice of two colors as it cannot be the same color as BG and cannot be the same color as MK. So 2 choices for GR.

TR →Once BG and GR are fixed we have 2 options for TR.

Therefore for MK, GR and TR we have 8 combinations.
D,A,B
D,A,D
D,B,A
D,B,D
A,B,A
A,B,D
A,D,A
A,D,B

Now we have to find how many options we have for BS. If TR, BG and RU are all in different colors then we have only one option for BS. However, if TR and RU have the same color then we have 2 options for BS.
Out of the 8 options above in 3 options TR and RU are in the same color (A) and in 5 options TR and RU are in different color.
Therefore, the total options are 24 x (5 x 1+ 3 x 2)=264

【Marking Scheme】

- Choosing 3 countries and fixing their colors → 5 pts
- Proper reasoning for the rest of the countries based on restrictions of already fixed colors → up to 10 pts (here partial credits will be given based on the progress)
- Finding the correct answer (even with guessing)→ 5 pts

Providing a working counting strategy (different from the solution) → up to 15 pts (based on the progress)

Answer: 264 ways