Individual Contest

Time limit: 120 minutes **2015/12/11**

Section A.

In this section, there are 12 questions. Fill in the correct answer on the space provided at the end of each question. Each correct answer is worth 5 points. Be sure to read carefully exactly what the question is asking.

1. Evaluate $M=\dfrac{1}{9-\sqrt{80}}-\dfrac{1}{\sqrt{80}-\sqrt{79}}+\dfrac{1}{\sqrt{79}-\sqrt{78}}-\cdots-\dfrac{1}{\sqrt{10}-3}$. 【**Submitted by *Bulgaria_FPMG***】

【***Solution***】

Since $\dfrac{1}{\sqrt{n}-\sqrt{n-1}}=\sqrt{n}+\sqrt{n-1}$ for all *n*, the expression is equal to $9+\sqrt{80}-\sqrt{80}-\sqrt{79}+\sqrt{79}+\sqrt{78}-\ldots-\sqrt{10}-3=6$.

***ANS*:** 6

2. Find the smallest positive integer *n* such that both $2n$ and $3n+1$ are squares of integers. 【**Submitted by *VIETNAM***】

【***Solution***】

Let *n* be such that $3n+1=x^2$ and $n=2y^2$ for some positive integers *x* and *y*. Then $x^2=6y^2+1$. It follows that *x* is odd and *y* is even. The smallest possible value of *y* is 2, and the corresponding value of *x* is 5. Hence the smallest possible value of *n* is 8.

***ANS*:** 8

3. How many different possible values of the integer *a* are there so that $||x-2|-|3-x||=2-a$ has solutions? 【**Submitted by *Bulgaria_SMG***】

【***Solution***】

The expression $||x-2|-|3-x||$ measures the difference between the distance from *x* to 2 and the distance from *x* to 3. If *x* is not between 2 and 3, the difference is always 1, so that we may have $a=1$. If *x* is between 2 and 3, the difference is less than 1, and takes only the integer value 0 at $x=2.5$. Hence we may also have $a=2$. These are the only 2 possible values.

ANS: 2

4. If $\sqrt{k-9}$ and $\sqrt{k+36}$ are both positive integers, what is the sum of all possible values of *k*? 【**Submitted by *VARNA***】

【***Solution***】

Let $n^2=k+36$ and $m^2=k-9$ then $n^2-m^2=45$ and $(n-m)(n+m)=45$. The divisors of 45 are 1, 3, 5, 9, 15 and 45. There are three options. If $n+m=45$ and $n-m=1$ then $k=493$. If $n+m=15$ and $n-m=3$ then $k=45$. If $n+m=9$ and $n-m=5$ then $k=13$. The desired sum is $493+45+13=551$.

***ANS*:** 551

5. Find the largest positive integer *n* such that the sum of the squares of the positive divisors of *n* is n^2+2n+2.【**Submitted by *Bulgaria_SMG***】

【***Solution***】

If *n* is prime, then the sum of squares of its divisors is n^2+1, which is too small. If *n* is the square of a prime, then the sum is n^2+n+1, 1 less than the desired value. If *n* is a higher power of a prime, the sum will be too large. Hence $n=ab$ where *a* and *b* are relatively prime integers greater than 1. Then the sum of the squares of its positive divisors is at least $n^2+a^2+b^2+1 \geq n^2+2ab+1=n^2+2n+1$. Thus *a* and *b* must be prime numbers so that they are the only other positive divisors. Moreover, we must have $1=a^2+b^2-2ab=(a-b)^2$ so that $a-b=1$. Thus the only possibility is $(a, b) = (3, 2)$, so that $n=6$ is the largest, and in fact, the only solution.

***ANS*:** 6

6. Find the smallest two-digit number such that its cube ends with the digits of the original number in reverse order.【**Submitted by *MALAYSIA***】

【***Solution***】

The cube of the number $10a+b$ is $(10a+b)^3=1000a^3+300a^2b+30ab^2+b^3$. Then the units digit of b^3 is *a*. If $b=1$, 2, 3, 4, 5, 6, 7, 8 or 9, then $a=1$, 8, 7, 4, 5, 6, 3, 2 or 9 and $30ab^2+b^3=31$, 968, 1917, 1984, 3875, 6696, 4753, 4352 or 22599 respectively. Thus the only possibility is 99.

***ANS*:** 99

7. A Mathematics test consists of 3 problems, each problem being graded independently with integer points from 0 to 10. Find the number of ways in which the total number of points for this test is exactly 21.【**Submitted by *ROMANIA***】

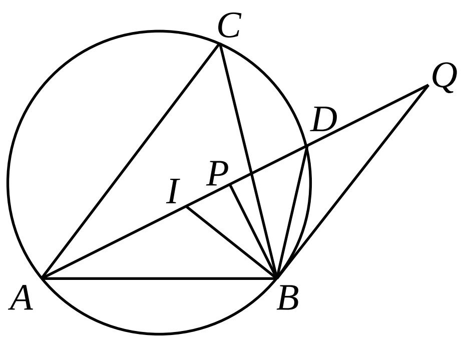
【***Solution***】

If the first problem is graded 10, then there are 10 ways to grade the other two problems in order to obtain 21 points $1+10=2+9=3+8=4+7=5+6=6+5=7+4=8+3=9+2=10+1=21$. If the first problem is graded 9, then there are 9 ways to grade the other two problems in order to obtain 21 points, and so on. If the first problem is graded 1, then there is 1 way to grade the other two problems in order to obtain 21 points. The total number of ways is thus: $10+9+8+\cdots+1=55$.

***ANS*:** 55 ways

8. In the triangle *ABC*, the bisectors of $\angle CAB$ and $\angle ABC$ meet at the in-center *I*. The extension of *AI* meets the circumcircle of triangle *ABC* at *D*. Let *P* be the foot of the perpendicular from *B* onto *AD*, and *Q* a point on the extension of *AD* such that $ID=DQ$.

Determine the value of $\dfrac{BQ \times IB}{BP \times ID}$.【***2013 AITMO Proposal***】



【***Solution***】

Let us prove that $\triangle BPQ$ is similar to $\triangle IBQ$.

First $\angle IAB = \angle CAD = \angle CBD$.

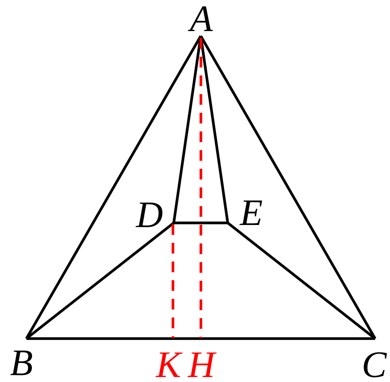
As $\angle IBA = \angle IBC$, we have $\angle IAB + \angle IBA = \angle CBD + \angle IBC$ so that $\angle DIB = \angle DBI$, thus $DI = DB = DQ$. This means $\triangle IBQ$ is a right triangle with $\angle IBQ = 90°$. As $\angle Q$ is a common angle, we thus have $\triangle BPQ$ is similar to $\triangle IBQ$.

Hence, $\frac{BQ}{BP} = \frac{IQ}{IB} = \frac{2ID}{IB}$. Consequently, $\frac{BQ \times IB}{BP \times ID} = 2$.

***ANS*: 2**

9. *D* and *E* are points inside an equilateral triangle *ABC* such that *D* is closer to *AB* than to *AC*. If $AD = DB = AE = EC = 7$ cm and $DE = 2$ cm, what is the length of *BC*, in cm? 【**Submitted by *Philippines***】

【***Solution***】



Let *H* and *K* be the respective feet of perpendicular from *A* and *D*. Let the side length of *ABC* be *x*. Then

$BH = \frac{x}{2}$, $BK = \frac{x}{2} - 1$ and $DK = \sqrt{7^2 - (\frac{x}{2} - 1)^2}$.

Furthermore, we have $AH = \frac{\sqrt{3}x}{2}$ and $AH - DK = 4\sqrt{3}$. From $\sqrt{49 - (\frac{x}{2} - 1)^2} = \frac{\sqrt{3}x}{2} - 4\sqrt{3}$, we have $-\frac{x^2}{4} + x = \frac{3x^2}{4} - 12x$, which yields $x = 13$.

***ANS*: 13 cm**

10. In a class, five students are on duty every day. Over a period of 30 school days, every two students will be on duty together on exactly one day. How many students are in the class? 【**Submitted by *ROMANIA***】

【Solution】

Let *n* be the number of students in the class and let *N* be the number of pairs of students that are on duty on each of the 30 days. Since there were 5 students on duty every day, $N = \binom{5}{2} \times 30 = 300$. On the other hand, since two students are on duty on exactly 1 days, $N = \binom{n}{2} = \frac{n \times (n-1)}{2}$. From $0 = n^2 - n - 600 = (n+24)(n-25)$, we have *n*=25.

Arrange the students in a 5×5 array as follows.

A	**B**	**C**	**D**	**E**
F	**G**	**H**	**I**	**J**
K	**M**	**N**	**O**	**P**
Q	**R**	**S**	**T**	**U**
V	**W**	**X**	**Y**	**Z**

We divide the 30 days into 6 blocks of 5 days each, with the following duty schedule.

Block 1	ABCDE	FGHIJ	KMNOP	QRSTU	VWXYZ
Block 2	AGNTZ	BHOUV	CIPQW	DJKRX	EFMSY
Block 3	AHPRY	BIKSZ	CJMTV	DFNUW	EGOQX
Block 4	AIMUX	BJNQY	CFORZ	DGPSV	EHKTW
Block 5	AJOSW	BFPTX	CGKUY	DHMQZ	EINRV
Block 6	AFKQV	BGMRW	CHNTX	DIOTY	EJPUZ

This structure is called a (30,25,6,5,1) block design, and is based on the finite affine geometry of order 5.

ANS: 25 students

11. A committee is to be chosen from 4 girls and 5 boys and it must contain at least 2 girls. How many different committees can be formed? 【**Submitted by *ROMANIA***】

【***Solution***】

There is 1 way to include all 4 girls, 4 ways to include 3 of them and 6 ways to include 2 of them. Thus the girls can be chosen in $1+4+6=11$ ways. The boy can be chosen in $2^5=32$ ways. Hence a total of $11\times 32=352$ different committee can be formed.

***ANS*:** 352 committee

12. Find the largest positive integer such that none of its digits is 0, the sum of its digits is 16 but the sum of the digits of the number twice as large is less than 20. 【**Submitted by *Russia***】

【***Solution***】

In adding the number to itself, there must be carries over as otherwise the sum of the digits of the number twice as large will be 32. A carry over reduces the sum of the digits by 9, so we need at least two of them. The largest such number is 55111111.

***ANS*:** 55111111

Section B.

Answer the following 3 questions. Show your detailed solution on the space provided after each question. Each question is worth 20 points.

1. What is the number of ordered pairs (x, y) of positive integers such that $\frac{3}{x}+\frac{1}{y}=\frac{1}{2}$ and $\sqrt{xy}\ge 3\sqrt{6}$? 【**Submitted by *Bulgaria_SMG***】

【***Solution***】

We have $\frac{3}{x}+\frac{1}{y}=\frac{1}{2}\Leftrightarrow 6y+2x=xy\Leftrightarrow (x-6)(y-2)=12$. (5 points)

So (x, y) = (7, 14), (8, 8), (9, 6), (10, 5), (12, 4) or (18, 3). (5 points)

The respective values of xy are 98, 64, 54, 50, 48 and 54. (5 points)

Since $(3\sqrt{6})^2=54$, we must reject (10, 5) and (12, 4), leaving behind 4 ordered pairs. (5 points)

Ans: 4

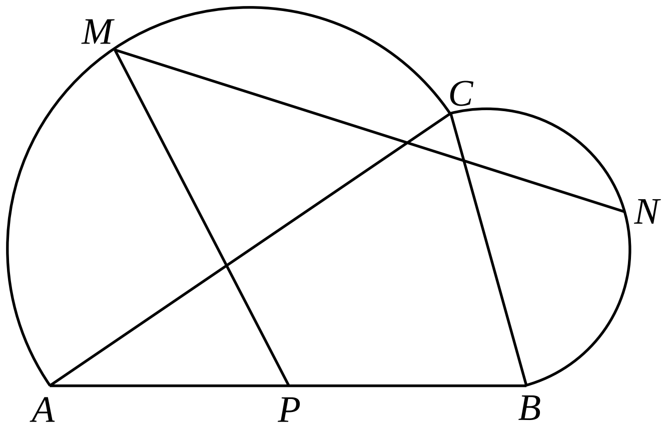
2. What is the minimum number of the 900 three-digit numbers we must draw at random such that there are always seven of them with the same digit-sum? 【**Submitted by *ROMANIA***】

【***Solution***】

The minimum digit-sum is 1 and the maximum digit sum is 27. Each of these appears once, in 100 and 999 respectively. Each of 2 and 26 appears three times, in 101, 110 and 200 in the former case and in 998, 989 and 899 in the latter case(5 points). Similarly, each of 3 and 25 appears six times. (5 points) Every other digit-sum appears at least seven times. We may draw $2\times 1+2\times 3+23\times 6=146$ numbers without having seven with the same digit-sum. However, if we draw another number, we are sure to have seven numbers with the same digit-sum. (10 points)

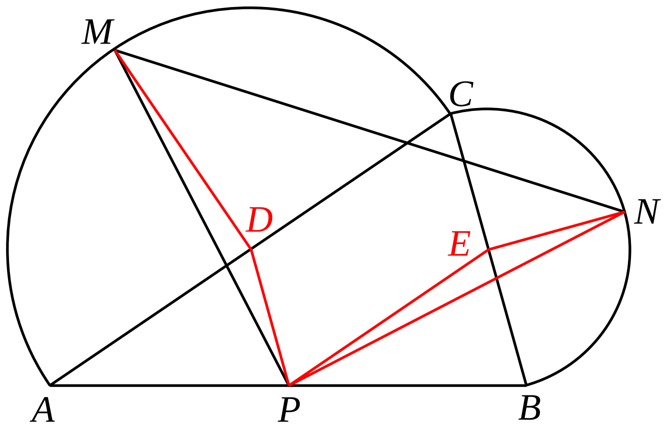
ANS: 147

3. Point M is the midpoint of the semicircle of diameter AC. Point N is the midpoint of the semicircle of diameter BC and P is midpoint of AB. Prove that $\angle PMN=45°$. 【**Submitted by *VARNA***】



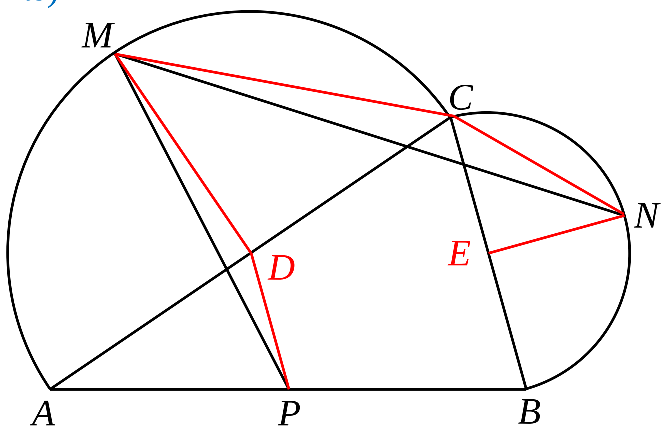
【***Solution* 1**】

Let points D and E be the midpoints of AC and BC respectively. Then $DM = DC = PE$ and $DP = CE = NE$.(5 points)Moreover, $\angle MDP = 90° + \angle ADP$ while $\angle PEN = 90° + \angle PEB$. Hence triangles MPD and PNE are congruent. (5 points) We have $MP = PN$ and $\angle PMD = \angle EPN$. It follows that $\angle MPN = \angle MPD + \angle DPE + \angle EPN = \angle MPD + \angle ADP + \angle PMD = 180° - 90° = 90°$, (5 points)so that $\angle PMN = \angle PNM = 45°$.(5 points)



【***Solution* 2**】

Construct the perpendicular line MD and NE, where D is the midpoint of AC and E is the midpoint of BC. Since $DM = DC$, we have $\angle DMC = \angle DCM = 45°$. Similarly, $\angle ECN = 45°$.(5 points)



D and P are the midpoints of AC and AB, respectively, so $DP // BC$ and $DP = \frac{1}{2}BC = CE$. Since $DP // BC$, $\angle ADP = \angle ACB$. (5 points) Thus we have $\angle MDP = 90° + \angle ADP = 45° + 45° + \angle ACB = \angle DCM + \angle ECN + \angle ACB = \angle MCN$.

And since $\frac{CM}{DM} = \sqrt{2} = \frac{CE}{CN} = \frac{DP}{CN}$, triangle MDP and MCN are similar triangles. (5 points) Hecnce $\angle PMD = \angle NMC$.

So $\angle PMN = \angle PMD + \angle DMN = \angle NMC + \angle DMN = \angle DMC = 45°$ (5 points)