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2014 Taiwan Selection Test for PMWC and EMIC

Intermediate Round Paper I Answer

1. In selling two cell-phones for 6000 dollars each, a merchant gains 20% in one sale and loses 20% in the other. What is the net gain or net loss?

【Solution】

The cost of the first cell phone is $6000 \div \frac{100}{100+20} = 5000$ dollars. The cost of the

second cell phone is $6000 \div \frac{100}{100-20} = 7500$ dollars. The net loss is $5000 + 7500 -$

$2 \times 6000 = 500$ dollars.

Answer : the net loss is 500 dollars

2. A red marker and a black marker are placed on two squares of an 8×8 chessboard which may not be rotated or reflected. How many different placements are there if the two markers may not be in the same row or in the same column?

【Solution】

There are $8 \times 8 = 64$ choices to place the first marker and $(8 - 1)(8 - 1) = 49$ choices to place the second marker. Hence the total number of placements is $64 \times 49 = 3136$.

Answer : 3136

3. A car goes uphill at 30 kph and then immediately returns downhill at 50 kph to its starting point. If the round trip takes 4 hours, how far, in km, has the car travelled uphill?

【Solution】

The amount of time spent going uphill is $4 \times \frac{50}{30+50} = 2.5$ hours. During this time,

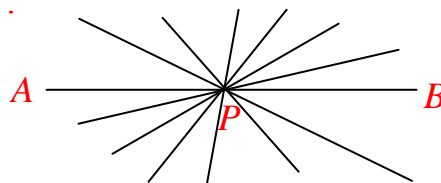
the car has travelled $2.5 \times 30 = 75$ km.

Answer : 75 km

4. Seven straight lines on the plane meet each others. The measure of the angle between any two lines is an integral number of degrees. What is the maximum measure, in degrees, of the smallest angle between two of the lines?

【Solution】

Move the seven lines meet at point P . The seven lines divide 180° into seven angles. Since $7 \times 25 < 180 < 7 \times 26$, the maximum measure of the smallest angle is at most 25° . This can be achieved if two of the seven angles have measures 25° while the other five have measures 26° .



Answer : 25°

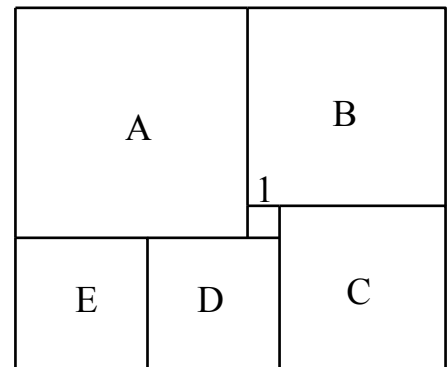
5. When Jerry was born, his sister's age was $\frac{1}{4}$ of his mother's age. Now his age is $\frac{1}{4}$ of the age of his mother while his sister's age is $\frac{1}{3}$ of his father's age. In nine more years, Jerry's age will be $\frac{1}{3}$ of the age of his father's age. How old is Jerry's sister now?

【Solution】

The least common multiple of 3 and 4 is 12. Suppose Jerry is 12 years old now. Then his mother is 48. When Jerry was born, she was 36, so that Jerry's sister was 9. Hence she is now 21, and Jerry's father is 63. The difference between three times Jerry's age and his father's age is $63 - 3 \times 12 = 27$ now, and is reduced by 2 every year. Hence Jerry's age will be $\frac{1}{3}$ of his father $27 \div 2 = 13.5$ years. Since the actual number of years for this to happen is 9, Jerry's sister is now $21 \times \frac{9}{13.5} = 14$ years old.

Answer : 14 years old

6. The diagram below shows a rectangle divided into six squares. Five of them are labelled A, B, C, D and E. If the side length of the sixth square is 1 cm, what is the area, in cm^2 , of the rectangle?



【Solution】

In going from D through C and B to A, the side-lengths increases by 1 cm each step. Hence the difference between the side lengths of A and D is 3 cm. Now D and E are of the same size, and the sum of their side lengths is 1 greater than the side length of A. If D is of side length 2 cm, then the side length of A will be $2 + 2 - 1 = 3$, and the difference in side length is $3 - 2 = 1$ cm. Since the actual difference is 3 cm, the side length of D is $2 + (3 - 1) = 4$, so that the side lengths of C, B and A are 5, 6 and 7 respectively. It follows that the area of the rectangle is $(7 + 6)(6 + 5) = 143 \text{ cm}^2$.

Answer : 143 cm^2

7. Three brothers have 99, 63 and 54 dollars respectively. They agree to share their wealth in the following way. Each step involves only two of them. The one with more money will give some to the other so that he will have twice as much as before. After three moves, what is the minimum value of the largest difference between the amounts of money of the brothers?

【Solution】

The total amount of money, which remains constant, is $99 + 63 + 54 = 216$ dollars. A fair share is 72 dollars. Working backwards to before the third move, one brother has $72 \div 2 = 36$ dollars, another has 72 dollars and the third $72 + 36 = 108$ dollars. Before the second move, the brother with 36 dollars should still have 36 dollars. Hence the one with 108 dollars has $108 \div 2 = 54$ dollars and the other one has $72 + 54 = 126$ dollars.

It is now clear that in the first step, the one with 63 dollars gets another 63 dollars from the one with $36+63=99$ dollars. It follows that the minimum difference is 0 dollars.

Answer : 0 dollars

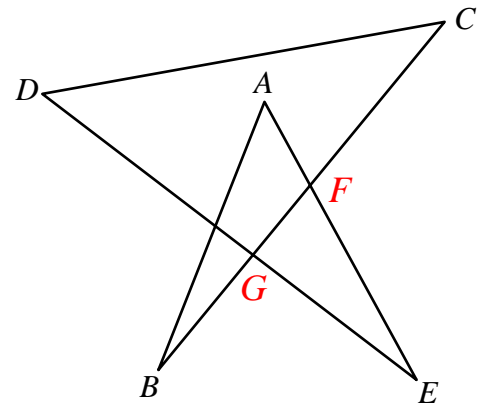
8. In the diagram below, determine $\angle A + \angle B + \angle C + \angle D + \angle E$, in degrees.

【Solution】

Let BC intersect AE at F and DE at G . Then

$$\begin{aligned} \angle A + \angle B + \angle C + \angle D + \angle E \\ = \angle GFE + \angle FGE + \angle FEG = 180^\circ \end{aligned}$$

Answer : 180°



9. What are the last two digits of the product of 2014 copies of 7?

【Solution】

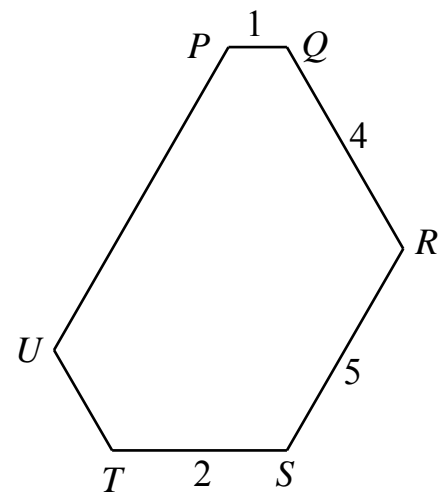
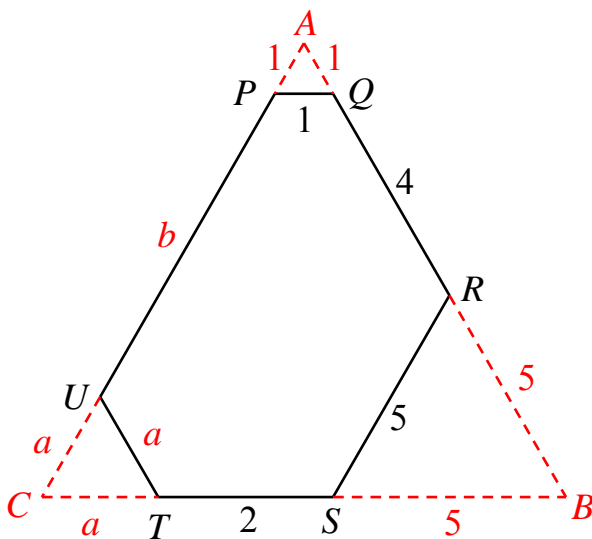
Note that $7^1 = 7$, $7^2 = 49$, $7^3 = 343$ and $7^4 = 2401$. The last two digits so far are 07, 49, 43 and 01. Note that $7^5 = 7(2400 + 1) = 16800 + 7$, so that the last two digits are 07 again. Similarly, the last two digits of $7^6 = 7(16800 + 7)$ is 49, and so on. So we have a cycle of length 4. When 2014 is divided by 4, the remainder is 2. Thus we will be at the second position in the cycle, and the last two digits of 7^{2014} are 49.

Answer : 49

10. $PQRSTU$ is a hexagon in which all angles are 120° . If $PQ = 1$ cm, $QR = 4$ cm, $RS = 5$ cm and $ST = 2$ cm, what is the perimeter, in cm, of $PQRSTU$?

【Solution】

Extend the sides of the hexagon to form an equilateral triangle ABC , as shown in the diagram below.



Then APQ , BRS and CTU are all equilateral triangles. Hence

$$BC = CA = AB = AQ + QR + RB = PQ + QR + RS = 10 \text{ cm.}$$

Now

$$TU = CT = BC - BS - ST = BC - RS - ST = 3 \text{ cm}$$

and

$$UP = CA - AP - CU = CA - PQ - CT = 6 \text{ cm.}$$

Thus the perimeter of $PQRSTU$ is $1+4+5+2+3+6=21$ cm.

Answer : 21 cm

11. The units digit of a positive integer is 2. If we move it to the other end, we get a new number which is twice the old number. What is smallest possible value of the old number?

【Solution】

In the diagram below, we carry out the long division of the new number by 2. The first digit of the new number is 2. So the first digit of the quotient is 1. Since the quotient is the old number, this 1 is the second digit of the new number. We continue until we hit 2 as the last digit of the quotient. The first time this happens is 1052, but the digit before the 2 is odd, but the new number must be even. Thus the smallest number we seek is 105263157894736842.

$\begin{array}{r} 105263 \\ 2 \overline{)210526} \\ \underline{2} \\ 10 \\ \underline{10} \\ 0 \\ \underline{0} \\ 10 \\ \underline{10} \\ 0 \\ \underline{0} \\ 5 \\ \underline{4} \\ 12 \\ \underline{12} \\ 0 \\ \underline{0} \\ 6 \\ \underline{6} \\ 0 \end{array}$	$\begin{array}{r} 157894 \\ 2 \overline{)315789} \\ \underline{2} \\ 11 \\ \underline{10} \\ 15 \\ \underline{14} \\ 17 \\ \underline{16} \\ 18 \\ \underline{18} \\ 0 \\ \underline{0} \\ 9 \\ \underline{8} \\ 1 \end{array}$	$\begin{array}{r} 736842 \\ 2 \overline{)1473684} \\ \underline{14} \\ 7 \\ \underline{6} \\ 13 \\ \underline{12} \\ 16 \\ \underline{16} \\ 8 \\ \underline{8} \\ 0 \\ \underline{0} \\ 4 \\ \underline{4} \\ 0 \end{array}$
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Answer : 105263157894736842

12. Find the smallest positive integer which leaves 64 as the remainder when divided by 150 and 51 as the remainder when divided by 151.

【Solution】

Note that when 151 is divided by 150, the remainder is 1. On the other hand, we have $150 \times 150 = (151 - 1)(151 - 1) = 151^2 - 2 \times 151 + 1$ so that when 22500 is divided by 151, the remainder is 1. Consider the number $22500 \times 51 + 151 \times 64 = 1157164$. When it is divided by 150, the remainder comes from the second term and is equal to 64. When it is divided by 151, the remainder come from the first term and is equal to 51. We can reduce it by any multiple of $150 \times 151 = 22650$. Since $1157164 = 22650 \times 51 + 2014$, the smallest number we seek is 2014.

Answer : 2014

2014 Taiwan Selection Test for PMWC and EMIC

Intermediate Round Paper II Answer

1. Find all two-digit numbers such that when multiplied by any positive one-digit number, its digit-sum does not change.

【Solution】

When a one-digit number is multiplied by 5, it becomes a 0 or a 5, with a carry over of up to 4. Hence no chain reaction of carry over can start. For 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, the change in digit-sum is 0, 4, -1, 3, -2, 2, -3, 1, -4 and 0 respectively. The -4 must go with the 4, so that 1 must go with 8 to yield 18 and 81. Similarly, the others are 27, 72, 36, 63, 45, 54, 90 and 99. In the diagram below, the entries in bold-face indicate changes in digit-sums. Hence only 18, 45, 90 and 99 have the desired property. (Each correct answer with detail 5 points)

\times	9	8	7	6	5	4	3	2
18	162	144	126	108	90	72	54	36
27	243	216	189					
36	324	288						
45	405	360	315	270	225	180	135	90
54	486							
63	567							
72	648							
81	729							
90	810	720	630	540	450	360	270	180
99	891	792	693	594	495	396	297	198

2. Put each of the nine numbers 1, 1, 5, 5, 5, 5, 10, 10 and 10 into a different square of a 3×3 table so that the sum of the three numbers in each row and in each column is different.

【Solution】

Suppose the three 10s are in different rows. Then the row-sums can only be 20, 16 and 12. Since they are different, each must appear. However, 16 and 12 cannot both appear. Hence the three 10s are not in different rows, and similarly not in different columns. By symmetry, we may assume that they are placed as shown in the diagram below on the left. This forces the placement of a 5 and a 1 as shown, again up to symmetry. The two squares in the same diagonal as a 10 must both be 5s, as otherwise the three numbers in the second row will have the same sum as the three numbers in the second column. If the second 1 is placed in the empty square on the second row, the sum 11 will appear twice. Hence the placement is unique up to symmetry, and is shown in the diagram below on the right.

10	10	5
10		
1		

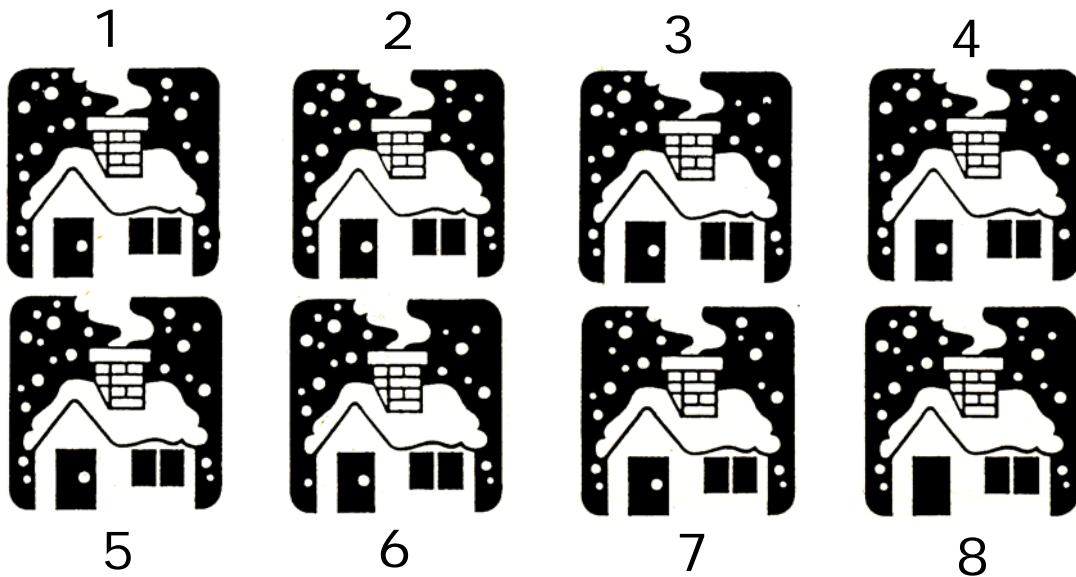
10	10	5
10	5	5
1	1	5

3. Boris picks at random a ten-digit number 7804320512. Anna picks five positive integers a, b, c, d and e , and Boris tells her the value of $78a + 4b + 32c + 5d + 12e$. What is the minimum number of times Anna has to do this to be able to deduce that Boris' number is 7804320512?

【Solution】

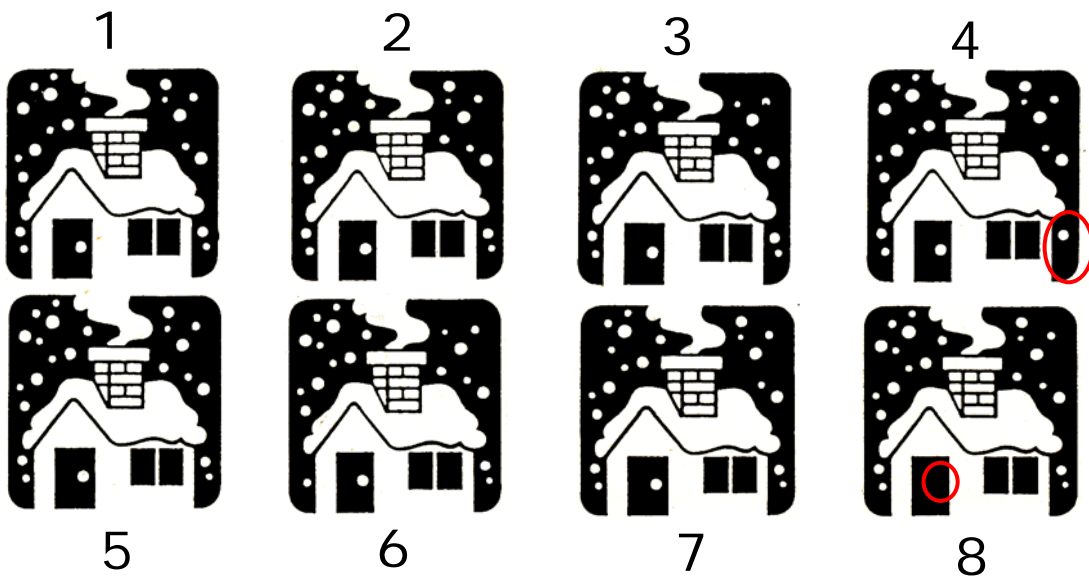
Anna only has to do this once, by picking $e = 1, d = 100, c = 10000, b = 1000000$ and $a = 100000000$. Boris' answer will be 7804320512.

4. Can you determine which two pictures are identical?

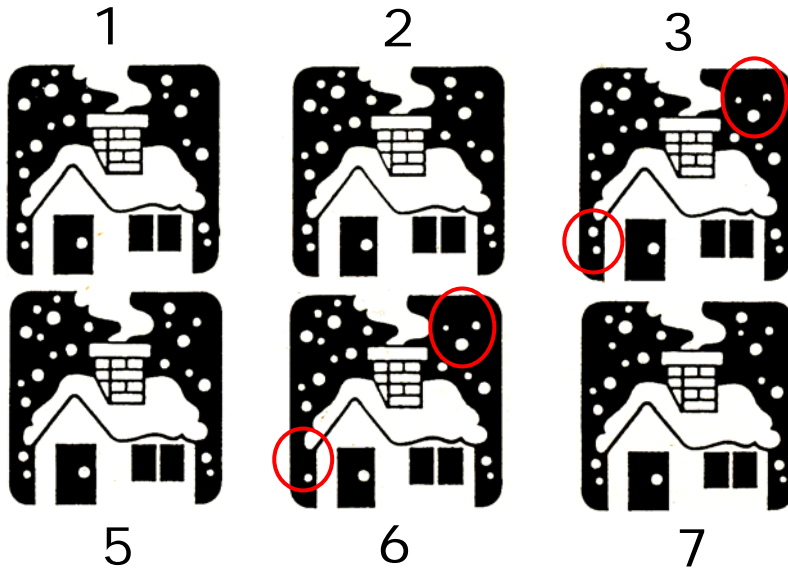


【Solution】

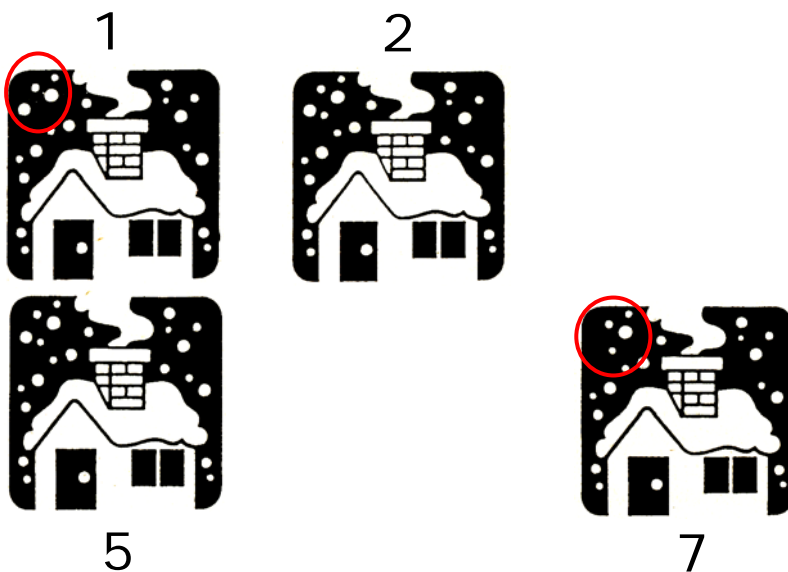
On figure No. 8, there are not point at the door, on figure No. 4 there are only one point at right lower corner, different with other figure, we may delete it.



On figure No. 3 and No. 6 there are only 3 points at right upper corner, but at left lower corner there are 1 point and 2 points respectively, hence we can delete those two figures too.



On figure No. 1 and No. 7 there are only 4 points at left upper corner, but the sides are different, hence we can delete those two figures.



The remaining two figure No. 2 and No. 5 are identical.

Answer : No. 2 and No. 5 are identical