2014 **Taiwan Selection Test for PMWC and EMIC**
Final Round Paper I Answer

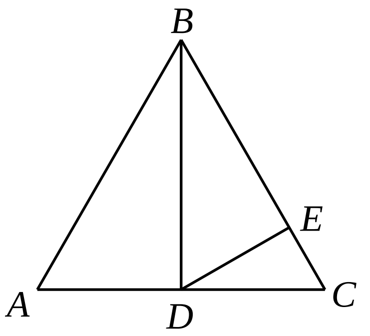
1. Going at the average speed of 40 km per hour, we will be 1 hour late. Going at the average speed of 60 km per hour, we will be 1 hour early. At what average speed, in km per hour, should we go in order to arrive just in time?

【Solution】

The slower speed is $\frac{1}{40}$ hour per km while the faster speed is $\frac{1}{60}$ hour per km. To arrive at the average time, our speed should be $(\frac{1}{40}+\frac{1}{60})\div 2=\frac{1}{48}$ hour per km, or 48 km per hour.

Answer: 48 km per hour

2. ABC is an equilateral triangle of side length 4 cm. D is a point on AC such that BD is perpendicular to AC, and E is a point on CB such that DE is perpendicular to CB. What is the area, in cm^2, of a square whose side length is DE?



【Solution】

Triangles BAD and BCD are congruent. Hence $AD = CD$ = 2 cm. By Pythagoras' Theorem, $BD=\sqrt{AB^2-AD^2}=2\sqrt{3}$ cm. The area of triangle ABC is $\frac{1}{2}AC\times BD$ cm^2 while the area of triangle CBD is $\frac{1}{2}CB\times DE$ cm^2. Since the area of CBD is half that of ABC, $DE=\frac{1}{2}BD=\sqrt{3}$ cm. The area of the square whose side length is DE is $\sqrt{3}\times\sqrt{3}=3$ cm^2.

Answer: 3 cm^2

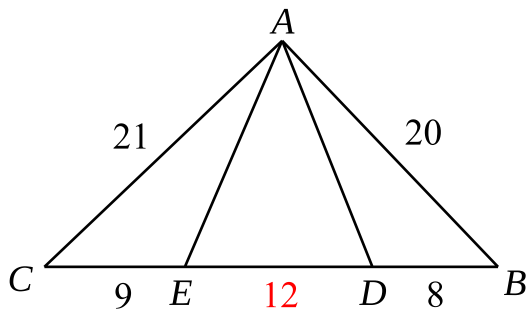
3. Trains go from town A to town B at regular intervals, all travelling at the same constant speed. A train going from town B to town A at the same constant speed along a parallel track meets the trains going in the opposite direction every 10 minutes. How often, in minutes, do the trains go from town A to town B?

【Solution】

Suppose the trains going from town A to town B all stop in their tracks while the train going from town B to town A continues to move. Since their combined speed is now reduced by half, the time between meeting one train and the next is now 20 minutes. Hence the trains go from town A to town B every 20 minutes.

Answer: 20 minutes

4. In triangle ABC, $BC = 29$ cm, $CA = 21$ cm and $AB = 20$ cm. D and E are points on BC such that $BD = 8$ cm and $CE = 9$ cm. Determine the measure, in degrees, of $\angle EAD$.



【Solution】

We have $DE = BC - BD - CE = 12$ cm. Note that $20^2 + 21^2 = 29^2$.

By the converse of Pythagoras' Theorem, $\angle CAB = 90°$.

Since $BE = BD + DE = BA$, $\angle BAE = \angle BEA$.

Since $CD = CE + ED = CA$, $\angle CAD = \angle CDA$.

Hence

$$\begin{aligned}\angle EAD &= 180° - \angle BEA - \angle CDA \\ &= 180° - \angle BAE - \angle CAD \\ &= 180° - (\angle BAD + \angle DAE) - (\angle CAE + \angle EAD) \\ &= 180° - (\angle BAD + \angle DAE + \angle CAE) - \angle EAD \\ &= 90° - \angle EAD\end{aligned}$$

It follows that $\angle EAD = 45°$.

Answer: $45°$

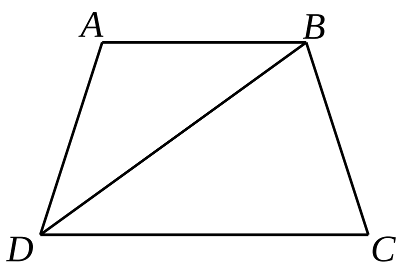
5. What is the largest possible remainder when a two-digit number is divided by the sum of its digits?

【Solution】

The digit-sum is at most 18, so that the remainder is at most 17. However, the only number with digit-sum 18 is 99, and the remainder when 99 is divided by 18 is only 9. For digit-sum 17, we have 98 and 89, and the respective remainders are 13 and 4. Hence the remainder is at most 15, and this is attained when 79 is divided by 16.

Answer: 15

6. In the quadrilateral $ABCD$, AB is parallel to DC, $AB < DC$ and $AD = BC$. The diagonal BD divides $ABCD$ into two isosceles triangles. Determine the measure, in degrees, of $\angle C$.



【Solution】

In triangle BAD, $\angle BAD > 90°$ since $AB < DC$ and AB is parallel to DC. Hence we must have $AB = AD$.

Now $\angle ADB = \angle ABD = \angle BDC = \frac{1}{2}\angle ADC$, and $\angle ADC = \angle BCD$. It follows that

$$\begin{aligned}\angle BCD &= 180° - \angle CBD - \angle BCD \\ &= 180° - \angle BCD - \frac{1}{2}\angle BCD\end{aligned}$$

Hene $\frac{5}{2}\angle BCD = 180°$ so that $\angle BCD = 72°$.

Answer: $72°$

7. Every two of A, B and C play one game against each other, scoring 2 points for a win, 1 point for a draw and 0 points for a loss. How many different pairs of numbers are there such that the first is A's total score and the second is B's total score?

【Solution】

Between them, A and B score at most 6 points and at least 2 points. If A and B have the same number of points, the desired pair is one of (3, 3), (2, 2) and (1, 1). Any of these can be achieved by having A and B draw the game between them, and get the remaining points off C. If A has more points than B, the desired pair is one of (4, 2), (4, 1), (4, 0), (3, 2), (3, 1), (3, 0), (2, 1) and (2, 0). Any of these can be achieved by having A beat B, and both get the remaining points off C. By symmetry, there are another 8 pairs in which B scores more points than A. Hence the total number of pairs is $3 + 2 \times 8 = 19$.

Answer: 19

8. Find the sum of all positive integers less than 100, each of which has exactly 10 positive divisors.

【Solution】

Consider any number with exactly 10 positive divisors. All its divisors may be listed in a 1×10 table or a 2×5 table because these are the only factorization of 10. If it is a 1×10 table, the number is the ninth power of a prime, but $2^9 = 512 > 100$. If it is a 2×5 table, the number is the product of the fourth power of a prime and a different prime. We have $2^4 \times 3 = 48$ and $2^4 \times 5 = 80$. There are no other values under 100 since $2^4 \times 7 = 112$ and $3^4 \times 2 = 162$. The sum we seek is 48+80=128.

Answer: 128

9. The minute hand of a clock is moving as though it is the hour hand, while the hour hand is moving as though it is the minute hand. At six o'clock in the evening, the clock is showing the correct time. Next day, shortly after seven o'clock in the morning, it shows the correct time again. How many minutes after seven o'clock does that happen?

【Solution】

At seven o'clock in the morning of next day, the minute hand has moved $390°$ and is pointing at 1. Meanwhile the hour hand has moved 13 times around and is pointing at 6 again. To get to the correct time, it had to make up $30°$. At the correct time, the hour hand, moving 12 times as fast as the minute hand, had moved $\frac{30°}{12-1}$. Since it is moving at the rate of $\frac{1}{2}°$ per minute, the number of minutes after seven o'clock was $\frac{30}{11} \div \frac{1}{2} = 5\frac{5}{11}$.

Answer: $5\frac{5}{11}$ minutes

10. The first three digits of a common multiple of 7, 8 and 9 form the number 523. What is the maximum value of the number formed from its last three digits?

【Solution】

The least common multiple of 7, 8 and 9 is $7 \times 8 \times 9 = 504$. The diagram below shows the long division of our number by 504.

$$\begin{array}{r} 10 \\ 504\overline{)523abc} \\ \underline{504} \\ 19ab \end{array}$$

Maximizing the number we seek is accomplished by maximizing the quotient. The first digit of the quotient must be 1 and the second one must be 0. The third one is at most 3 and the fourth one at most 9. Hence we take the quotient to be 1039. Note that $3 \times 504 = 1512$ and $9 \times 504 = 4536$. Hence the incomplete number on the fourth row is $1512 + 453 = 1965$ and the incomplete number on the second row is 523656. The maximum value of the number we seek is 656.

Answer: 656

11. Among the positive integers between 1000 and 10000, how many multiples of 9 are there such that the sum of the first two digits is equal to the sum of the last two digits?

【Solution】

A number is divisible by 9 if and only if the sum of its digits is divisible by 9. Hence the equal sum in the problem can only be 9 or 18. For 18, the only number is 9999. For 9, the first two digits form one of the numbers 90, 81, 72, 63, 54, 45, 36, 27 and 18, while the last two digits also form one of these numbers unless they form 09. Hence the total number of such multiples of 9 within range is $1 + 9 \times 10 = 91$.

Answer: 91

12. The numbers 1, 2, . . . , 25 are to be placed in a 5×5 table, with one number exactly in each square. Consecutive numbers occupy squares with a common side. Three of the numbers have been placed, as shown in the diagram below. Find the number of different placements of the other 22 numbers.

19		13		
		1		

【Solution】

The two squares adjacent to the square containing 19 must contain 18 and 20. If the square containing 18 is on the left edge, then the square containing 20 will be blocked off by a chain connecting the squares containing 13 and 18. Hence 18 and 20 must be placed as shown in the diagram below on the left, and the placement of 17 and 21 are also forced. Now the two unoccupied squares adjacent to the square containing 13 must contain 12 and 14. As before, they must be placed as shown in the diagram below in the middle, and the placement of 16, 22 and 15 are also forced.

19	18	13		
20	17			
21				
		1		

19	18	13	12	
20	17	14		
21	16	15		
22				
		1		

19	18	13	12	11
20	17	14	9	10
21	16	15	8	7
22		2	3	6
	24	1	4	5

Now the top right corner square must contain 11. The placement of 10, 9, 8, 7, 6, 5, 4, 3 and 2 are forced. The unoccupied square adjacent to the square containing 1 must contain 24. The remaining two squares can hold 23 and 25 in either order. Thus the number of different placements is 2.

Answer: 2

2014 **Taiwan Selection Test for PMWC and EMIC**

Final Round Paper II Answer

1. Insert three plus or minus signs between the digits of 123456789 so that the value of the resulting expression is 100.

【Solution】

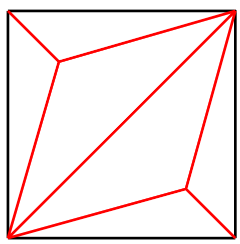
The three signs partition the 9 digits into four numbers. By the Pigeonhole Principle, we must have at least one number with at least 3 digits. If one of them have 4 or more digits, the numerical value of the expression will be way too large. This is also the case if there are two 3-digit numbers, because their hundreds digit must differ by at least 3. Hence exactly one 3-dight number, and it must be 123 for the same reason. The remaining numbers are 45, 67 and 89, and we have 123 − 45 − 67 + 89 = 100.

Answer: $123-45-67+89=100$

2. What is the minimum number of obtuse triangles into which a square may be dissected?

【Solution】

A dissection line must pass through each vertex of the square as otherwise a right triangle will result. Each side of the square must belong to a different triangle, so that there is at least 4 of them. If there are exactly 4, then the dissection lines mentioned earlier must all meet at a point. Four angles are formed there, with sum $360°$. Hence at most 3 of them can be obtuse. Suppose there are 5 triangles. The sum of their interior angles is $5\times180°=900°$. The angles at the vertices of the square account for $360°$. The remaining $540°$ must come from points inside the square. We may have a point which is a vertex of every triangle to which it belongs, accounting for $360°$. The remaining $180°$ arises from another point which lies on a side of one of the triangles. The first point may serve as the vertex of at most 3 obtuse angles while the second point may serve as the vertex of at most 1 obtuse angle. Since we have 5 triangles, not all of them can be obtuse. It follows that we must have at least 6 obtuse triangles. The diagram below shows that this is possible.



Answer: 6

3. There are 10 coins in each of 11 bags. Every coin in 10 of the bags is real, while every coin in the remaining bag looks real but is fake. All real coins weigh the same, and all fake coins also weigh the same. The two weights are unequal. We wish to determine which bag contains the fake coins, but we do not have to find out whether a fake coin is heavier or lighter than a real one. We have a balance which indicates the difference in weight between the contents of its two pans. What is the minimum number of weightings required?

【Solution】

Clearly, two weightings are necessary. We now show that two weightings are also sufficient. Label the bags 0 to 10. In the first weighing, put all 10 coins in bag 10 in the left pan and put 1 coin from each of the other 10 bags in the right pan. The difference in weight will not be 0 gm. By symmetry, we may assume that the 10 coins in the left pan are heavier, and for definiteness say by 10 gm. If the fake coins are in bag 10, then each fake coin is 1 gm heavier than a real one. If not, each fake coin is 10 gm lighter than a real one. In the second weighing, add to the left pan 9 coins from bag 9, 8 coins from bag 8, and so on, down to 1 coin from bag 1. Add all the remaining coins to the right pan. Observe that there are 55 coins in each pan. The results are summarized in the diagram below. Since all the results are different, we can identify the bag containing the fake coins.

The fake coins are in bag	The contents of the left pan are
0	100 gm heavier
1	80 gm heavier
2	60 gm heavier
3	40 gm heavier
4	20 gm heavier
5	neither heavier or lighter
6	20 gm lighter
7	40 gm lighter
8	60 gm lighter
9	80 gm lighter
10	10 gm lighter

Answer: 2

4. Two copies of each of 1, 2, . . . , 7 are to be placed in a 1×14 table, with exactly one number in each square. The number in the first square is 6, and the number in the last square is 1. There is exactly 1 other number between the two copies of 1, exactly 2 other numbers between the two copies of 2, 3 others between the 3s, 4 others between the 4s, 5 others between the 5s, 6 others between the 6s and 7 others between the 7s. Find all possible placements of the other 12 numbers.

6													1

【Solution】

Let the 14 squares be shaded alternately as shown in the diagram below. Note that the placements of the other copy of 6 and the other copy of 1 are forced.

6													1

One copy of each of 2 and 4 is in a shaded square while the other is in an un-shaded square. The two copies of each of 3, 5 and 7 are either both in shaded squares or both in un-shaded squares. There are three cases.

Case 1. Both copies of 7 are in shaded squares.

There is only one way to place the 7s, and two ways to permute the 2 and 4 in shaded squares. The placement of the other 2 and 4 are forced, shown in the diagram below. It is not possible to place both the 3s and the 5s in the un-shaded squares.

6	7	2	4		2		6	4	7		1		1
6	7		2		4	2	6		7	4	1		1

Case 2. Both copies of 5 are in shaded squares.

There is only one way to place the 5s, and two ways to permute the 2 and 4 in shaded squares. The placement of the other 4 is forced, but there are two choices for the placement of the other 2 in one instance. These are shown in the diagram below. It is not possible to place both the 3s and the 7s in the unshaded squares.

6	4	2	5		2	4	6		5		1		1
6	4		5		2	4	6	2	5		1		1
6	2		5	2	4		6		5	4	1		1

Case 3. Both copies of 3 are in shaded squares.

There are two ways to place the 3s. In the first way, there are two ways to permute the 2 and 4 in shaded squares. The placement of the other 4 is forced, but there are two choices for the placement of the other 2 in one instance. These are shown in the diagram below. It is not possible to place both the 5s and the 7s in the un-shaded squares.

6	3		4		3	2	6	4	2		1		1
6	3		4		3		6	4	2		1	2	1
6	3		2	4	3	2	6		4		1		1

The second way leads to the unique placement of all 14 digits.

6	2	7	4	2	3	5	6	4	3	7	1	5	1