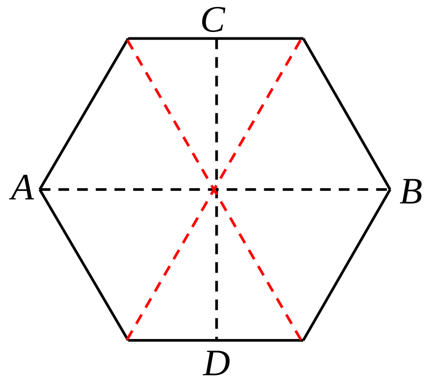
2015 Taiwan Selection Test for PMWC and EMIC

Final Round Paper I (Time Allowed : 90 Minutes)

- Write down all answers on the answer sheet. Each problem is worth 10 points and the total is 120 points.

1. A and B are opposite vertices of regular hexagon. C and D are midpoints of opposite sides such that CD is perpendicular to AB. The area of the hexagon is 126 cm^2. What is the area, in cm^2, of the rectangle with length AB and width CD?

【First Solution】

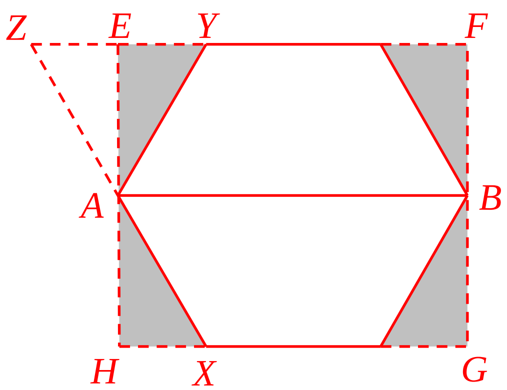


As shown in the diagram, the hexagon consists of six equilateral triangles whose base is $\frac{AB}{2}$ and whose height is $\frac{CD}{2}$. Hence its area is

$$6\times\frac{1}{2}\times\frac{AB}{2}\times\frac{CD}{2}=\frac{3}{4}\times AB\times CD.$$

It follows that the area of the rectangle is $126\div\frac{3}{4}=168\text{ cm}^2$.

【Second Solution】



Label additional points as shown in the diagram. We seek the area of the rectangle $EFGH$. Rotate triangle AHX about A to AEZ. Then AYZ is an equilateral triangle whose area is $\frac{1}{6}$ that of the hexagon. Hence the total area of triangles AHX and AEY is $\frac{1}{6}\times 126=21\text{cm}^2$. Similarly, the total area of the other two shaded triangles is also 21 cm^2. It follows that the area of $EFGH$ is $126+21+21=168\text{ cm}^2$.

Answer：168 cm^2

2. The six-digit number $\overline{17A32B}$ is a multiple of 88. What is the maximum value of the quotient when this number is divided by 88?

【Solution】

The six-digit number is a multiple of 8 as well as a multiple of 11. This means that $\overline{32B}$ must be a multiple of 8. Hence $B=0$ or 8. Now $(7+3+B)-(1+A+2)=7+B-A$ must be a multiple of 11. If $B=0$, then $A=7$. If $B=8$, then $A=4$. The maximum value of the quotient is $177320\div 88=2015$.

Answer：2015

3. The weight of each of five coins has three possible values. However, there are only two different values among the weights of the five coins. In how many different ways can this happen?

【Solution】

We may have either 4 coins of one weight and the fifth coin of a different weight, or 3 coins of one weight and the other two coins of a different weight. The number of ways of choosing the coins is $5+\frac{5\times 4}{2}=15$. The number of ways of choosing the weights is $3\times 2=6$. Hence the total number of ways is $15\times 6=90$.

Answer： 90 ways

4. Going with the current, a ship takes 6 hours to get from A to B. Going against the current, the ship takes 7 hours to get from B back to A. The speeds of the ship and of the current do not change. In how many hours will a raft flow downstream from A to B?

【Solution】

Let the distance between A and B be 1 unit. Then the speed of the ship plus the speed of the current is $\frac{1}{6}$ while the speed of the ship minus the speed of the current is $\frac{1}{7}$. Hence the speed of the current is $(\frac{1}{6}-\frac{1}{7})\div 2=\frac{1}{84}$, so that it will take the raft 84 hours to flow downstream from A to B.

Answer：84 hours

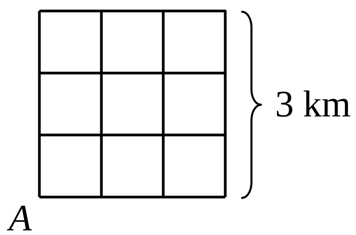
5. A garden is in the shape of a right triangle. The length of a side of the right angle is 35 m. The lengths, in m, of the other two sides are also positive integers. What is the minimum value of the perimeter, in m, of this garden?

【Solution】

By Pythagoras' Theorem, 35^2 is the difference of the squares of the lengths, in m, of the other two sides. Hence it is the product of the sum of and the difference between the lengths, in m, of the other two sides. The former is minimum when the latter is maximum. Now 49×25 is the decomposition of 35^2 into two factors which are as close to each other as possible without being equal. Hence the minimum value of the sum of the other two sides is 49 m, and the minimum value of the perimeter of the garden is $35+49=84$ m.

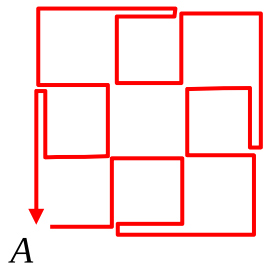
Answer：84 m

6. The diagram shows a small town with 9 blocks each measuring 1 km by 1 km. Starting from a corner of the town, a man sweeps every section of the streets and returns to his starting point. He may walk along sections he has already swept. What is the minimum distance, in km, he has to move?



【Solution】

There are eight intersections on the edges of the town which are not corners. Each is the junction of three street sections. The man must walk over one of them which he has already swept. It follows that to minimize the total distance he has to move, he must go over twice each of the 4 street sections joining two of those eight intersections. Since the total number of street sections is 24, the minimum distance is $24+4=28$ km. The diagram below shows a way to do so.



Answer：28 km

7. The factorial of a positive integer *n*, denoted by *n*!, is the product of all positive integers from 1 to *n* inclusive. Thus $5!=1\times2\times3\times4\times5$. Find the largest three-digit number which is equal to the sum of the factorials of its three digits.

【Solution】

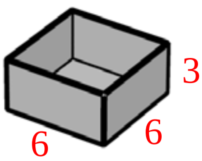
Note that 1!=1, 2!=2, 3!=6, 4!=24, 5!=120, 6!=720 and 7!=5040. Hence all three digits are less than 7. Actually, they must be less than 6 since 6! starts with 7. If all three digits are less than 5, then the number is at most $3\times4!=72$, which is not a three-digit number. Hence at least one digit is 5. Not all three digits are 5s. If there is another 5, then our number is at least $2\times5!=240$. Hence the first digit must be 2, but $2!+5!+5!\neq255$. Hence there is exactly one 5. Our number is at least 1!+1!+5!=127 and at most 4!+4!+5!=168. Hence the first digit is 1 and our number is at most 1!+4!+5!=145. Hence 145 is the largest three-digit number with the desired property. In fact, it is the only such number.

Answer：145

8. The surface area of a box without a lid is 108 cm^2. What is the maximum value of its volume, in cm^3?

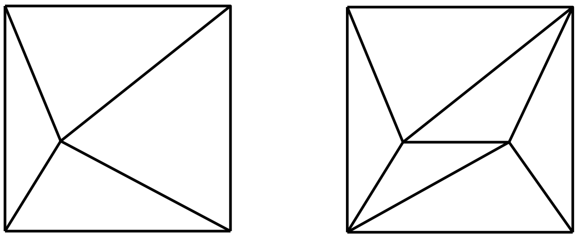
【Solution】

If we fit two copies of this open box together, we have a closed box whose surface area is $2\times106=216$ cm^2. Its volume is maximum when the dimensions of the box are as close to one another as possible. Since $216=6^3$, the closed box with maximum volume is the $6\times6\times6$ box. Hence the open box with minimum volume is the $3\times6\times6$ box with volume $3\times6\times6=108$ cm^3.



Answer：108 cm^3

9. Inside a square are 15 points. Some pairs of these points are joined by line segments, and some of these points are joined to some of the vertices of the squares by line segments. All line segments do not intersect except at their endpoints. They divide the square into regions each of which is bounded by exactly three segments. How many triangles are there? The diagram below on the left shows that if there is only 1 point inside the triangle, then there are 4 triangles. The diagram below on the right shows that if there 2 points inside the square, then there are 6 triangles.



【Solution】

The sum of the angles at the four vertices of the square is $4\times 90^\circ = 360^\circ$. The sum of the angles at each of the 15 points is 360°. It follows that the sum of the angles of the triangles is $16\times 360^\circ$. Since the sum of the angles of each triangle is 180°, the number of triangles is $16\times 360^\circ \div 180^\circ = 32$.

Answer：32 triangles

10. In a chess tournament, each participant plays a game against every other participant. Two of them are Grade 7 students while the remaining ones are all Grade 8 students. A win is worth 2 points, a draw 1 point and a loss 0 points. Between them, the two Grade 7 students score 16 points, while each Grade 8 student has the same score as one another. What is the maximum number of Grade 8 student in this tournament?

【Solution】

Of the 16 points scored by the two Grade 7 students, 2 points come from the game between themselves. Thus 14 points are won against Grade 8 students. Since each Grade 8 student has the same score as one another, the Grade 7 students score the same number of points against them. Hence the number of Grade 8 students must be a divisor of 14, and the maximum value is 14. This can be attained if all games among Grade 8 students are draws, and each Grade 8 student beats one Grade 7 student and draws with the other.

Answer：14 students

11. Each of the first two terms of a sequence is 59. Starting from the third term, each is the sum of the preceding two terms. What is the remainder when the 2015th term of this sequence is divided by 3?

【Solution】

We may replace each number in the sequence by the remainder when it is divided by 3. Thus the beginning of the sequence is 2, 2, 1, 0, 1, 1, 2, 0, 2, 2, …. With the reappearance of two 2s in a row, we see that the sequence repeats in a cycle of length 8, namely (2, 2, 1, 0, 1, 1, 2, 0). Since $2015 = 8\times 251 + 7$, the 2015th remainder is the seventh in the cycle, namely 2.

Answer：2

12. Start with a square piece of paper. In the first move, use a straight cut to divide it into two pieces. In each subsequent move, use a straight cut to divide any of the pieces into two pieces. What is the minimum number of moves required in order to obtain at least five 15-sided polygons among the pieces?

【First Solution】

We first show that 59 cuts are sufficient. Use the first 4 cuts to divide the square piece of paper into five quadrilateral pieces. Use 11 cuts on each piece to trim corners and produce a 15-sided polygon. We now prove that 59 cuts are necessary. Consider the sum of the angles of all the polygons. Initially, this sum is $360°$ since we only have one square. Each cut increases this sum by at most $180°$ at each endpoint of the cut. Suppose five polygons each with exactly 15 sides have been obtained by n cuts. Then the sum of the angles of all $n+1$ polygons is at most $(n+1)\times 360°$, while the sum of the angles of the five polygons is $5\times(15-2)\times 180°$. Hence the sum of the angles of the other polygons is at least $(n+1-5)\times 180°$. It follows that $5\times 13+(n-4)\le 2n+2$, which yields $n\ge 59$.

【Second Solution】

We first show that 59 cuts are sufficient. Use the first 4 cuts to divide the square piece of paper into five quadrilateral pieces. Use 11 cuts on each piece to trim corners and produce a 15-sided polygon. We now prove that 59 cuts are necessary. Consider the total number of sides of all the polygons. Initially, this total is 4 since we only have one square. Each cut increases this total by 2 and by at most 1 at each endpoint. Suppose five polygons each with exactly 15 sides have been obtained by n cuts. Then the total number of sides of all $n+1$ polygons is at most $4n+4$, while the total number of sides of the five polygons is $5\times 15=75$. Hence the total number of sides of the other polygons is at least $3(n+1-5)=3n-12$. It follows that $75+3n-12\le 4n+4$, which yields $n\ge 59$.

Answer：59 moves

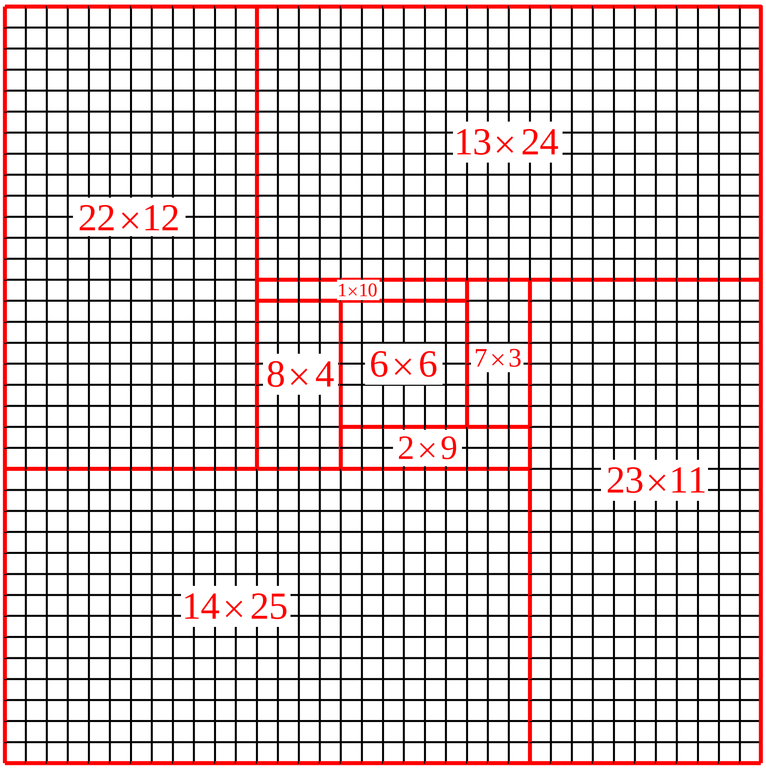
2015 Taiwan Selection Test for PMWC and EMIC
Final Round Paper II (Time Allowed : 60 Minutes)

- Each question is worth 25 marks for a maximum score of 100 marks. Fill in the answers in the space provided. Detailed solutions to Problems 2 and 4 are required.

1. Dissect a square into 9 rectangles such that no two adjacent rectangles can be combined into a single rectangle by erasing their common border.

【Solution】

The diagram below shows a dissection of a 36×36 square into 9 rectangles. Their respective dimensions are 22×12, 13×24, 1×10, 7×3, 23×11, 8×4, 6×6, 2×9 and 14×25. No 2 of the 18 dimensions are equal. Hence no 2 of the rectangles, whether adjacent or not, cannot be combined into a single rectangle even with rearrangements.



2. There are at least 30 real coins among 31 coins, and the last may be real or fake. All real coins weigh the same. A fake coin is either heavier or lighter than a real coin. We wish to determine whether there is a fake coin, and if so, whether it is heavier or lighter. What is the minimum number of weighings required on an standard balance?

【Solution】

Suppose we perform only one weighing and there is no equilibrium, we know that there is a fake coin, but we do not know whether it is heaver or lighter. Thus two weighings are necessary. We now show that two weighings are sufficient. In the first weighing, place 14 coin in each pan. We consider two cases.

Case 1. There is equilibrium.

We know that these 28 coins are all real. In the second weighing, place 3 of them in one pan, and place the 3 coins left off the first weighing in the other pan. If we have equilibrium, there is no fake coin. If there is no equilibrium, we can tell whether the fake coin is heavier or lighter because we know in which pan it must be.

Case 2. There is no equilibrium.

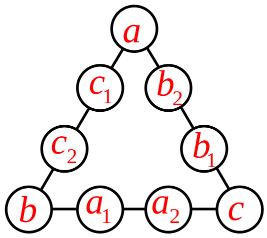
We know that one of these 28 coins is fake. Place 7 of the coins on the heavier side in one pan and the remaining 7 on the other pan. If there is equilibrium, the fake coin is not among these 14, and must therefore be lighter. If there is no equilibrium, the fake coin is among the 14 from the first weighing, and must therefore be heavier.

Answer：2

3. In the diagram below, replace each of a, a_1, a_2, b, b_1, b_2, c, c_1 and c_2 with a different one of the numbers from 1 to 9, so that $a>b>c$, $a_1>a_2$, $b_1>b_2$, $c_1>c_2$ and $b+a_1+a_2+c=c+b_1+b_2+a=a+c_1+c_2+b=20$. Find all of different solutions.

【Solution】

Since $a+b+c+a_1+a_2+b_1+b_2+c_1+c_2=45$, $a+b+c=15$, so that a, b and c are the numbers in any row, column or diagonal of the magic square in the diagram below.

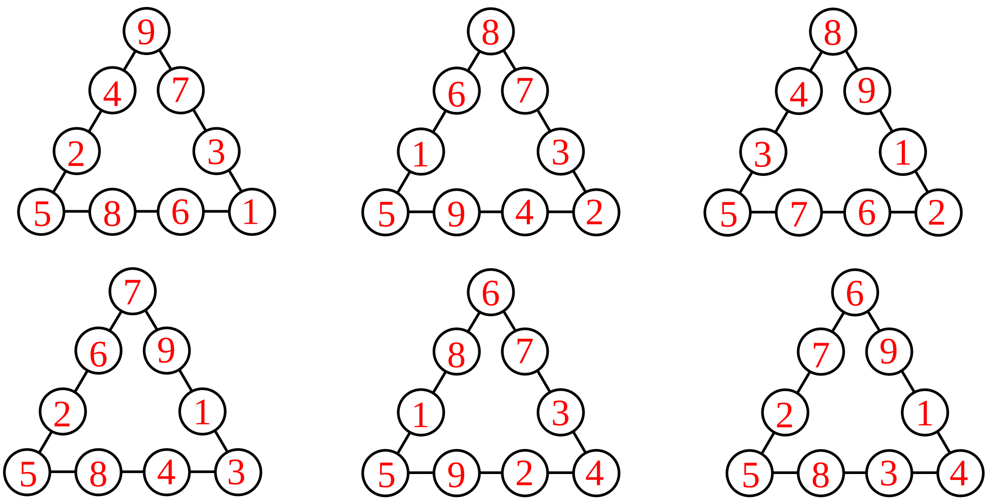


8	1	6
3	5	7
4	9	2

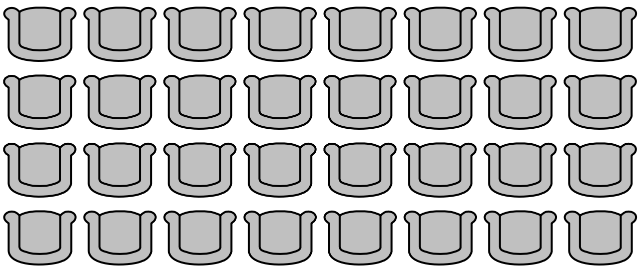
Now $a_1+a_2=20-(b+c)=a+20-(a+b+c)=a+5$. Similarly, $b_1+b_2=b+5$ and $c_1+c_2=c+5$. The following chart shows that there are six different ways. These are marked in boldface.

a	b	c	$\{a_1、a_2、b_1、b_2、c_1、c_2\}$	a_1+a_2	b_1+b_2	c_1+c_2
9	4	2	{1, 3, 5, 6, 7, 8}	6 + 8		1 + 6
8	6	1	{2, 3, 4, 5, 7, 9}	4 + 9		2 + 4
8	4	3	{1, 2, 5, 6, 7, 9}	6 + 7	2 + 7	
7	6	2	{1, 3, 4, 5, 8, 9}		3 + 8	3 + 4
					2 + 8	
9	5	1	{2, 3, 4, 6, 7, 8}	**6 + 8**	**3 + 7**	**2 + 4**
					4 + 6	
				4 + 9	**3 + 7**	**1 + 6**
8	5	2	{1, 3, 4, 6, 7, 9}		4 + 6	
				6 + 7	**1 + 9**	**3 + 4**
					2 + 8	
7	5	3	{1, 2, 4, 6, 8, 9}	**4 + 8**	**1 + 9**	**2 + 6**
					4 + 6	
				2 + 9	**3 + 7**	**1 + 8**
6	5	4	{1, 2, 3, 7, 8, 9}		2 + 8	
				3 + 8	**1 + 9**	**2 + 7**

The six solutions are:

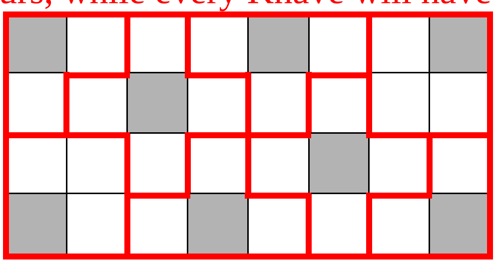


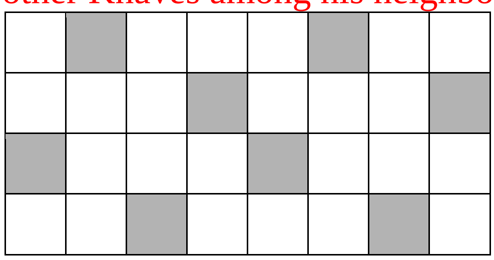
4. At a conference, 32 people are seated in 4 rows of 8. Each is either a Knight who always tells the truth, or a Knave who always lies. Each claims that there are at least one Knight and at least one Knave among the people occupying seats adjacent to his in the same row or the same column. What is the minimum number of Knaves at this conference?



【Solution】

We divide the 32 seats into 8 groups as shown in the diagram below on the left. We claim that among the people occupying seats in the same group, there is at least one Knave. If all of them are Knights, then the one occupying the shaded seats will have only Knights as his neighbours, and this Knight makes a false claim, which is a contradiction. It follows that the number of Knaves is at least 8. We may have as few as 8 Knaves, if they occupy the shaded seats in the diagram below on the right. Then every Knight will have at least one Knight and at least one Knave among his neighbours, while every Knave will have no other Knaves among his neighbours.





Answer：8