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2015 Taiwan Selection Test for PMWC and EMIC
Intermediate Round Paper I (Time Allowed : 90 Minutes)

- Write down all answers on the answer sheet. Each problem is worth 10 points and the total is 120 points.
- 1. A passenger fell asleep when the train is halfway to its destination. He woke up when the remaining part of the journey is equal to half the distance covered by the train while he was asleep. What fraction of the journey was he asleep?

【Solution】

We may assume that the remaining distance is 1 unit. Then he has slept for 2 units.

Hence the whole journey is 6 units and the fraction we seek is $\frac{2}{6} = \frac{1}{3}$

Answer : $\frac{1}{3}$

2. Two clocks both show 9:00 am. When the first clock shows 10:00 am, the second clock shows 10:04 am. Both clocks run at constant speed. When the first clock shows 1:30 pm on the same day, what time is shown on the second clock?

【Solution】

The speeds of the two clocks are in the ratio $60:64=15:16$. When the first clock shows 1:30 pm, it has gone $4.5 \times 60 = 270$ minutes. In the meantime, the second clock has gone $270 \times \frac{16}{15} = 288$ minutes. Hence the time it shows is 1:48 pm.

Answer : 1:48 pm

3. At most how many acute angles can a convex polygon have?

【Solution】

If an angle of a convex polygon is acute, then the corresponding exterior angle is obtuse. Now the sum of all the exterior angles of a convex polygon is 360° . Hence the number of acute angle is at most 3. This maximum is attained by any acute triangle.

Answer : 3 acute angles

4. A bakery sells three types of pies, meat pies, fruit pies and plain pies. It has 15 kinds of meat pies, 42 kinds of fruit pies and 25 kinds of plain pies. Kerry wants to buy 2 pies of different types. How many different choices can he make?

【Solution】

If Kerry does not buy meat pies, he has $42 \times 25 = 1050$ choices. If he does not buy fruit pies, he has $25 \times 15 = 375$ choices. If he does not buy plain pies, he has $15 \times 42 = 630$ choices. Thus Kerry has $1050 + 375 + 630 = 2055$ different choices.

Answer : 2055 different choices

5. Each physics book costs \$50, each chemistry book costs \$75 and each biology book costs \$100. There are 60 books in a shipment. The total cost of the physics books is equal to the total cost of the chemistry books, which is greater than the total cost of the biology books. Moreover, the number of physics books is odd. How many biology books are there in this shipment?

【Solution】

The ratio of the numbers of physics and chemistry books is 3: 2, and the ratio of the numbers of physics and biology books is greater than 2:1. Since the total number of book is 60 and the number of physics books is odd, the actual numbers of physics and chemistry books may be 33 and 22, in which case the number of biology books is 5. The next set of values is 27, 18 and 15, but the number of biology books is already more than half the number of physics books. It follows that the number of biology books can only be 5.

Answer : 5 biology books

6. Each of A, B, C and D plays a game against each of the other three. There are no draws. A, B and C have the same number of wins, and A beat D. How many wins does D have?

【Solution】

The total number of games is $\frac{4 \times 3}{2} = 6$. Since A beats D, each of A, B and C has 1 or 2 wins. Since A beats D, D does not have 3 wins. Hence each of A, B and C has 2 wins and D has 0 wins.

Answer : 0 wins

7. There are three piles of beads, containing respectively 11, 7 and 6 beads. In each move, we may choose two piles and double the number of beads in the smaller pile by transferring the appropriate number of beads from the larger pile. What is the minimum number of moves required to obtain three piles each containing 8 beads?

【Solution】

Working backwards, the numbers of beads in the three piles before the last step must be (4, 8, 12). Before the second last step, the numbers may be (2, 10, 12), (2, 8, 14), (4, 4, 16) or (4, 6, 14). Only (4, 6, 14) may be obtained from (6, 7, 11). Hence three steps are both necessary and sufficient. Specifically, double 7 from 11 in the first step, double 6 from 14 in the second step and double 4 from 12 in the third step.

Answer : 3 moves

8. In a four-digit even number, none of the digits is 6, 7, 8 or 9. What is the sum of all such numbers?

【Solution】

There are 5 choices for the thousands digit, 6 choices for each of the hundreds and tens digit and only 3 choice for the units digit.

The sum of all the units digit is $(0 + 2 + 4) \times 5 \times 6 \times 6 = 1080$.

The sum of all the tens digits is $(0 + 1 + 2 + 3 + 4 + 5) \times 5 \times 6 \times 3 \times 10 = 13500$.

The sum of all the hundreds digits is $13500 \times 10 = 135000$.

The sum of all the thousands digits is $(1 + 2 + 3 + 4 + 5) \times 6 \times 6 \times 3 \times 1000 = 1620000$.

The grand total is $1620000 + 135000 + 13500 + 1080 = 1769580$.

Answer : 1769580

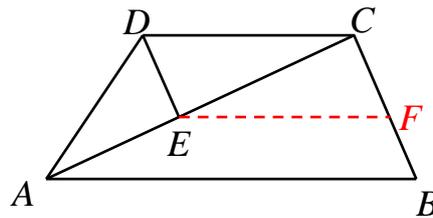
9. Wendy has been walking for 3.5 hours. In any one-hour period, she covers exactly 5 km. What is the maximum value of her speed, in km per hour?

【Solution】

In the first 3 hours, Wendy covers exactly $3 \times 5 = 15$ km. In the last 0.5 hour, she can cover at most another 5 km. Hence her speed is at most $\frac{40}{7} = 5\frac{5}{7}$ km per hour. This can be attained if she alternately covers 5 km each half hour and rest for half an hour.

Answer : $\frac{40}{7} = 5\frac{5}{7}$ km

10. In the quadrilateral $ABCD$, AB is parallel to CD . The line through D parallel to BC intersects AC at E . If the areas of triangles ADE and CDE are 60 cm^2 and 80 cm^2 respectively, what is the area, in cm^2 , of $ABCD$?



【Solution】

If we consider AE and CE as the respective bases of triangles ADE and CDE , then the triangles have a common height. Hence $\frac{CE}{CA} = \frac{80}{60+80} = \frac{4}{7}$. Draw the line through E parallel to AB , intersecting BC at F . Then $CDEF$ is a parallelogram so that CDE and CFE have the same area. It follows that the area of triangle ABC is $80 \times \frac{7^2}{4^2} = 245 \text{ cm}^2$. Hence in cm^2 , the area of $ABCD$ is $60 + 80 + 245 = 385$.

Answer : 385 cm^2

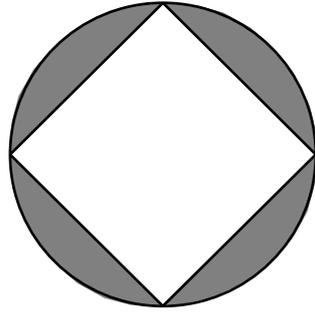
11. What is the maximum number of positive integers we can choose among the first 2015 such that the difference between any two chosen numbers is not a prime number?

【Solution】

Classify the first 2015 positive integers by their remainders when divided by 4. There are 504 numbers in each class except one, which has 503. If we choose all 504 numbers of one class, the difference between any two of them is a multiple of 4, and therefore cannot be a prime number. We now show that if we choose a particular positive integer, we can choose at most one of them. Clearly, the numbers which are 2, 3, 5 or 7 greater cannot be chosen. Among the number 1, 4 and 6 larger, their pairwise differences are 2, 3 and 5, so that we can choose at most one of them. It follows that we cannot choose more than one quarter of the numbers, so that 504 is indeed maximum.

Answer : 504 positive integers

12. Each vertex of a square lies on a circle. The part of the circle outside the square has area 224 cm^2 . What is the radius, in cm, of the circle if we take $\pi = \frac{22}{7}$?



【Solution】

The area of the square is equal to 2 times the square of the radius of the circle. The area of the circle is equal to π times the square of the radius of the circle.

Hence the square of the radius of the circle is equal to $224 \div (\frac{22}{7} - 2) = 196 \text{ cm}^2$. It

follows that the radius of the circle is 14 cm.

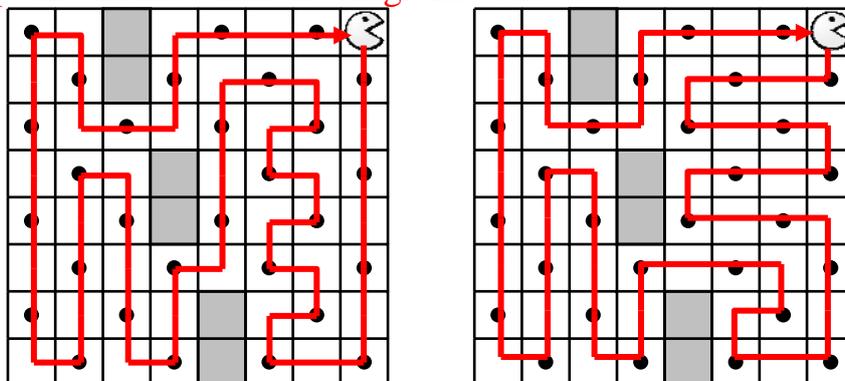
Answer : 14 cm

2015 Taiwan Selection Test for PMWC and EMIC Intermediate Round Paper II (Time Allowed : 60 Minutes)

- Each question is worth 25 marks for a maximum score of 100 marks. Fill in the answers in the space provided. Detailed solutions to Problems 2 and 4 are required.
1. The diagram below shows the 8×8 playing board of a video game. The six shaded squares are impassible. Some squares are marked with dots. Starting from the top left corner, Miss Pacman moves from square to adjacent square on the same row or the same column. As she passes a square with a dot, she eats the dot. Find a path which she can follow in order to eat all the dots and then return to her starting square, without passing through any square more than once.

【Solution】

Two possible paths are shown in the diagrams below.



2. There are at least 9 real coins among 10 coins, and the last may be real or fake. All real coins weigh the same. A fake coin is either heavier or lighter than a real coin. We wish to determine whether there is a fake coin, and if so, whether it is heavier or lighter. What is the minimum number of weighings required on a standard balance?

【Solution】

Suppose we perform only one weighing, and there is no equilibrium, we know that there is a fake coin, but we do not know whether it is heavier or lighter. Thus two weighings are necessary. We now show that two weighings are sufficient. In the first weighing, place 4 coins in each pan. We consider two cases.

Case 1. There is equilibrium.

We know that these 8 coins are all real. In the second weighing, place 2 of them in one pan, and place the 2 coins left off the first weighing in the other pan. If we have equilibrium, there is no fake coin. If there is no equilibrium, we can tell whether the fake coin is heavier or lighter because we know in which pan it must be.

Case 2. There is no equilibrium.

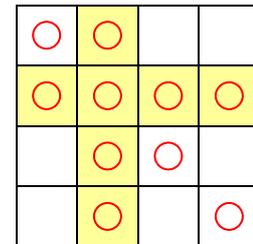
We know that one of these 8 coins is fake. Place 2 of the coins on the heavier side in one pan and the remaining 2 on the other pan. If there is equilibrium, the fake coin is not among these 4, and must therefore be lighter. If there is no equilibrium, the fake coin is among these 4, and must therefore be heavier.

Answer : 2

3. We wish to place 10 identical counters on different squares of a 4×4 chessboard such that the number of counters in each row and in each column is even, the following figure is an example. Find the number of different solutions.

【Solution】

If each row has at most 2 counters, then the total number of counters is at most 8. Hence there is a row with 4 counters. Similarly, there is a column with 4 counters. Hence each of the remaining rows and columns must have exactly two counters. Now there are 4 choices for the row with 4 counters, and 4 choices for the column with 4 counters. Ignoring this row and this column, we are left with a 3×3 subboard on which there is exactly one counter in each row and in each column. The counter on the first row has 3 choices for its column, the counter on the second row 2 choices and the counter on the third column 1 choice. Hence the total number of solutions is $4 \times 4 \times 3 \times 2 \times 1 = 96$. The diagram below shows one of these solutions.

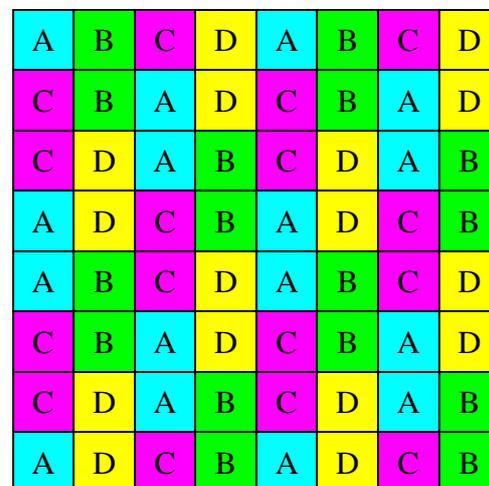
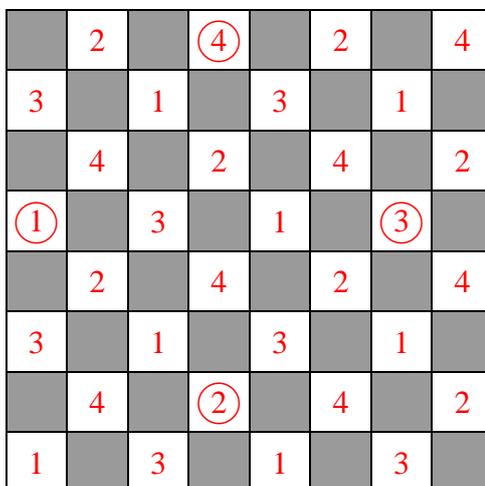


Answer : 96

4. In an 8×8 chessboard, the distance between two squares is defined as the difference between their row numbers plus the difference between their column numbers, always subtracting the smaller number from the larger. Each square is to be painted in one colour, and two squares at a distance 6 apart must be painted in different colours. What is the minimum value of colours required?

【Solution】

In the usual black and white pattern of the chessboard, two squares at a distance 6 apart are either both white or both black. In the diagram below on the left, we use numbers to show a 4-colouring of the white squares which satisfies the conditions. Moreover, the four squares containing a circled number are at the distance 6 from one another, so that 4 colours are necessary. If we fold the chessboard about its horizontal axis and copy the colours to the black squares, we obtain the 4-colouring as shown in the diagram below on the right. Combining the two yields the desired complete 4-colouring.



Answer : 4