2015 Taiwan Selection Test for PMWC and EMIC

Preliminary Round Paper II (Time Allowed : 90 Minutes)

- Each question is worth 25 marks for a maximum score of 300 marks. Write down all answers on the answer sheet. Each problem is worth 25 points and the total is 300 points.

1. Each of two four-digit numbers consists of four different digits. What is the maximum value when the smaller number is subtracted from the larger one?

【Solution】

The largest such four-digit number is 9876 and the smallest is 1023. Their difference is $9876-1023=8853$.

Answer：8853

2. A family has seven daughters. Each one after the first is two years younger than the one born before. If the eldest daughter is three times as old as the youngest, how old is the eldest?

【Solution】

The difference in age between the eldest and the youngest daughters is $2\times(7-1)=$ 12. This is double the age of the youngest daughter, who is $12\div 2=6$ years old. Hence the eldest daughter is $3\times 6=18$ years old.

Answer：18 years old

3. What is the sum of all the digits in the first 9999 positive integers?

【First Solution】

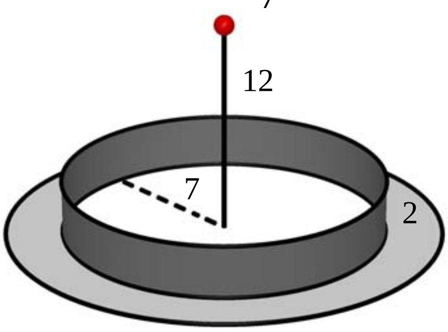
We can include the number 0. The sum of the digits of each of the pairs (0, 9999), (1, 9998), (2, 9997), ⋯, (4999, 5000) is 36. Hence the desired sum is $36\times 5000=180000$.

【Second Solution】

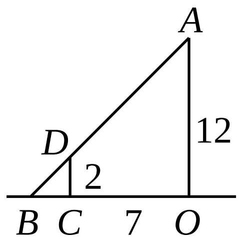
We can include the number 0, and add leading 0s to obtain 10000 four-digit numbers. In each digit, one-tenth of them are 0s, one tenth are 1s, and so on. Hence the desired sum is $4\times\frac{10000}{10}\times(0+1+2+3+4+5+6+7+8+9)=180000$.

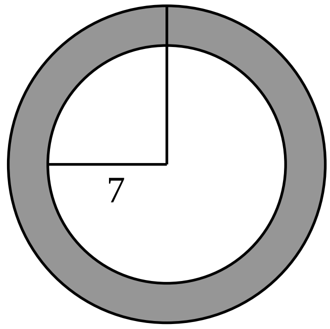
Answer：180000

4. The radius of a round enclosure is 7 m. It is surrounded by a fence of height 2 m along its circumference. A lamp is on top of a lamp post of height 12 m, which is at the centre of the enclosure What is the area, in m^2, of the shadow cast on the ground outside the enclosure if we take $\frac{22}{7}$)



【Solution】





Let O be the centre of the enclosure and A be where the lamp is. Let C be an arbitrary point of the circumference of the enclosure, and let D be the point on the top of the fence above C. Let B be the image of D under projection from A. Then triangles OAB and OCD are similar.

Hence $\frac{BO}{BC}=\frac{BC+CO}{BC}=\frac{AO}{CD}=\frac{12}{2}=6$ and CO=5 BC.

Since $OC=7\,\text{m}$, $BC=\frac{7}{5}\,\text{m}$. The area of the enclosure is $\pi\times 7^2=154\,\text{m}^2$. The area of the enclosure plus the shadow is $\pi\times(7\frac{7}{5})^2=221.76\ \ \text{m}^2$. Hence the area of the shadow is $221.76-154=67.76\ \ \text{m}^2$.

Answer：$\frac{1694}{25}=67\frac{19}{25}=67.76\ \ \text{m}^2$

5. In how many ways can Adam, Betty and Carol share 12 apples, with each getting at least one?

【First Solution】

We can divide 12 apples into three piles in the following ways: (10, 1, 1), (9, 2, 1), (8, 3, 1), (8, 2, 2), (7, 4, 1), (7, 3, 2), (6, 5, 1), (6, 4, 2), (6, 3, 3), (5, 5, 2), (5, 4, 3) and (4, 4, 4). There are $3\times2\times1=6$ ways to assign the piles to the three people if no two piles are equal in size. If exactly two piles are equal in size, the number of ways is 3. If all three piles are equal in size, the number is 1. It follows that the total number of ways is $6\times7+3\times4+1\times1=55$.

【Second Solution】

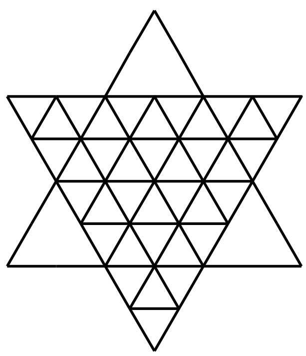
Adam can take 1 to 10 apples. If he takes 10, Betty can only take 1. If Adam takes 9 apples, Betty can take 1 or 2. If Adam takes 8 apples, Betty can take 1, 2 or 3, and so on. The total number of ways is $1+2+3+\cdots+10=55$.

【Third Solution】

Put the 12 apples in a row. Choose two of the gaps between apples and put a marker in each. This can be done in $\frac{11\times10}{2}=55$ ways. Adam will get all the apples to the left of the first marker, Betty all the apples between the markers and Carol all the apples to the right of the second marker.

Answer：55 ways

6. All triangles in the diagram are equilateral. Some of them overlap others. How many different triangles are there?



【Solution】

Let the side length of the smallest equilateral triangle be 1. Then there are 36 triangles of this size, 21 of them upside down and 15 right side up. There are 24 triangles of side length 2, 15 of them upside down and 9 right side up. There are 14 triangles of side length 3, 10 of them upside down and 4 right side up. There are 9 triangles of side length 4, 6 of them upside down and 3 right side up. There are 3 triangles of side length 5, all upside down. Finally, there are 2 triangles of side length 6, one of each kind. Hence the total is $36+24+14+9+3+2=88$.

Answer：88 triangles

7. The 100000 tickets for an event are numbered from 00000 to 99999. If a number contains two adjacent digits which differ by exactly 5, it wins a door prize. How many door prizes will be needed if all tickets are sold?

【Solution】

Let us count the number of tickets which do not win door prizes. The first digit can be any of 0, 1, ⋯, 9. Each subsequent digit must not differ from the preceding one by 5, and there are 9 choices. Hence there are $10\times 9^4=65610$ such tickets. The number of door prizes needed is $100000-65610=34390$.

Answer：34390 door prizes

8. Of 8 coins, 7 are known to be real and have the same weight. The other one may also be real, but may be a fake coin which is either heavier or lighter than a real coin. We want to know if there is a fake coin. If so, we wish to know whether it is heavier or lighter, but it is not necessary to identify the fake coin. What is the minimum number of weighing on a balance that would accomplish the task?

【Solution】

One weighing is not sufficient. If not all coins are involved, one of those left out may be a fake coin. If all coins are involved and there is no equilibrium, we only know that there is a fake coin, but not whether it is heavier or lighter than a real coin. Two weighings are sufficient. All coins are involved in the first weighing. If there is equilibrium, we know that all 8 coins are real. If not, we know that there is a fake coin. We now weigh 2 coins from the heavier side in the first weighing against the other 2 coins from the same side. If there is equilibrium, the fake coin is lighter than a real coin. If not, the fake coin is heavier than a real coin.

Answer：2 weighings

9. Dick goes to school by bicycle, riding at the same constant speed every day. One day, $\frac{3}{4}$ of the way to school, the bicycle breaks down, and he has to walk the rest of the way at a constant speed. If the amount of time Dick takes to go to school that day is twice the normal amount, how many times is his riding speed compared to his walking speed?

【First Solution】

Let the distance between Dick's home and school be 4 units. At double his normal amount of time, he can ride for a distance of $2\times 4=8$ units. On that day, he rides for $4\times\frac{3}{4}=3$ units and walks for $8-3=5$ units. It follows that riding 1 unit takes the same amount of time as walking 5 units, so that his riding speed is 5 times his walking speed.

Answer：5 times

10. The prices for each goose egg, chicken egg and quail egg are \$20, \$4 and \$2 respectively. Dee spends \$400 and buys 100 eggs, with at least one egg of each kind. Of the numbers of eggs of each kind that she buys, two are equal. How many chicken eggs has Dee bought?

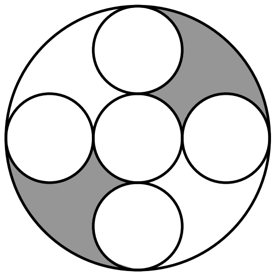
【Solution】

The average cost of each egg is \$4, which is the price of each chicken egg. Since the price of a goose egg is \$16 above the average and the price of a quail egg is \$2 less than the average, the number of quail eggs is 8 times the number of goose eggs. If the number of chicken eggs is equal to the number of quail eggs, then the total number of eggs must be a multiple of $1+8+8=17$, which is not the case. Hence the number of chicken eggs is equal to the number of goose eggs, and indeed $1+1+8=$ 10 is a divisor of 100. The number of chicken eggs Dee has bought is $100 \div 10 = 10$.

Answer：10 chicken eggs

11. The area of each of five circles is 133 cm^2. They are arranged in the form of cross inside a circle whose radius is three times as large. What is the total area, in cm^2, of the shaded parts in the diagram, taking $\frac{22}{7}$?

【Solution】



The area of the large circle is $3^2=9$ times the area of a small circle. By symmetry, each of the four parts of the large circle outside the small circles has the same area as a small circle. Hence the total area of the shaded parts is $2 \times 133 = 266\,\text{cm}^2$.

Answer：266 cm^2

12. Each of 100 boxers has different strength, and in any match, the stronger boxer always wins. How many matches are needed to determine the strongest boxer and the second strongest one?

【Solution】

To find the strongest boxes, we need to eliminate the other 99 boxers, and this requires 99 matches. Since $2^6=64<100<2^7=128$, we can organize these 99 matches as a seven-round knockout tournament with byes. Thus the strongest boxer has eliminated at most 6 other boxers on his way to the championship, and the second strongest boxes must be among them. It will take another 6 matches to find him, bringing the total number of matches to 105.

Answer：105 matches