International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2008.**

1. A standard 8×8 chessboard is modified by varying the distances between parallel grid lines, so that the cells are rectangles which are not necessarily squares, and do not necessarily have constant area. The ratio between the area of any white cell and the area of any black cell is at most 2. Determine the maximum possible ratio of the total area of the white cells to the total area of the black cells.

2. Space is dissected into non-overlapping unit cubes. Is it necessarily true that for each of these cubes, there exists another one sharing a common face with it?

3. A two-player game has $n>2$ piles each initially consisting of a single nut. The players take turns choosing two piles containing numbers of nuts relatively prime to each other, and merging the two piles into one. The player who cannot make a move loses the game. For each n, determine the player with a winning strategy, regardless of how the opponent may respond.

4. In the quadrilateral $ABCD$, AD is parallel to BC but $AB \neq CD$. The diagonal AC meets the circumcircle of triangle BCD again at A' and the circumcircle of triangle BAD again at C'. The diagonal BD meets the circumcircle of triangle ABC again at D' and the circumcircle of triangle ADC again at B'. Prove that the quadrilateral $A'B'C'D'$ also has a pair of parallel sides.

5. In the infinite sequence $\{a_n\}$, $a_0 = 0$. For $n \geq 1$, if the greatest odd divisor of n is congruent modulo 4 to 1, then $a_n = a_{n-1} + 1$, but if the greatest odd divisor of n is congruent modulo 4 to 3, then $a_n = a_{n-1} - 1$. The initial terms are 0, 1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 3, 2 and 1. Prove that every positive integer appears infinitely many times in this sequence.

6. $P(x)$ is a polynomial with real coefficients such that there exist infinitely many pairs (m, n) of integers satisfying $P(m)+P(n)=0$. Prove that the graph $y=P(x)$ has a centre of symmetry.

7. A contest consists of 30 true or false questions. Victor knows nothing about the subject matter. He may write the contest several times, with exactly the same questions, and is told the number of questions he has answered correctly each time. How can he be sure that he will answer all 30 questions correctly

   (a) on his 30th attempt;

   (b) on his 25th attempt?

Note: The problems are worth 4, 6, 6, 6, 8, 9 and 5+5 points respectively.