

注意：

允許學生個人、非營利性的圖書館或公立學校合理使用本基金會網站所提供之各項試題及其解答。可直接下載而不須申請。

重版、系統地複製或大量重製這些資料的任何部分，必須獲得財團法人臺北市九章數學教育基金會的授權許可。

申請此項授權請電郵 ccmp@seed.net.tw

Notice:

Individual students, nonprofit libraries, or schools are permitted to make fair use of the papers and its solutions. Republication, systematic copying, or multiple reproduction of any part of this material is permitted only under license from the Chiuchang Mathematics Foundation.

Requests for such permission should be made by e-mailing Mr. Wen-Hsien SUN ccmp@seed.net.tw

**International Mathematics
TOURNAMENT OF THE TOWNS**

Junior A-Level Paper

Fall 2009.

1. Ten jars contain varying amounts of milk. Each is large enough to hold all the milk. At any time, we can tell the precise amount of milk in each jar. In a move, we may pour out an exact amount of milk from one jar into each of the other 9 jars, the same amount in each case. Prove that we can have the same amount of milk in each jar after at most ten moves.
2. Mike has 1000 unit cubes. Each has 2 opposite red faces, 2 opposite blue faces and 2 opposite white faces. Mike assembles them into a $10 \times 10 \times 10$ cube. Whenever two unit cubes meet face to face, these two faces have the same colour. Prove that an entire face of the $10 \times 10 \times 10$ cube has the same colour.
3. Find all positive integers a and b such that $(a + b^2)(b + a^2) = 2^m$ for some integer m .
4. Let $ABCD$ be a rhombus. P is a point on side BC and Q is a point on side CD such that $BP = CQ$. Prove that the centroid of triangle APQ lies on the segment BD .
5. We have n objects with weights $1, 2, \dots, n$ grams. We wish to choose two or more of these objects so that the total weight of the chosen objects is equal to the average weight of the remaining objects. Prove that
 - (a) if $n + 1$ is a perfect square, then the task is possible;
 - (b) if the task is possible, then $n + 1$ is a perfect square.
6. On an infinite chessboard are placed 2009 $n \times n$ cardboard pieces such that each of them covers exactly n^2 cells of the infinite chessboard. The cardboard pieces may overlap. Prove that the number of cells of the infinite chessboard which are covered by an odd number of cardboard pieces is at least n^2 .
7. Olga and Max visited a certain Archipelago with 2009 islands. Some pairs of islands were connected by boats which run both ways. Olga chose the first island on which they land. Then Max chose the next island which they could visit. Thereafter, the two took turns choosing an accessible island which they had not yet visited. When they arrived at an island which was connected only to islands they had already visited, whoever's turn to choose next would be the loser. Prove that Olga could always win, regardless of the way Max played and regardless of the way the islands were connected.

Note: The problems are worth 4, 6, 6, 6, 2+7, 10 and 14 points respectively.

Solution to Junior A-Level Fall 2009

1. First Solution by Han Hyung Lee:

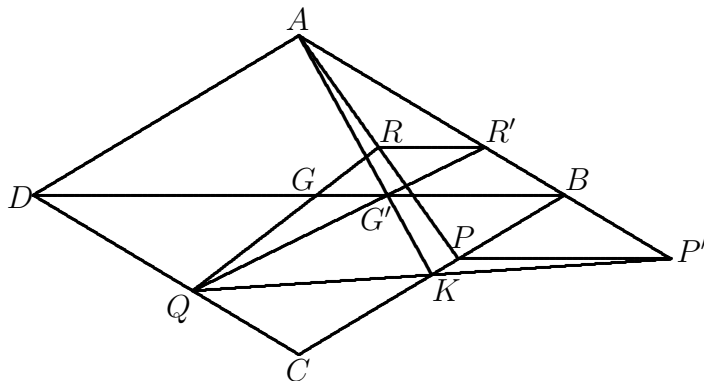
Pour from each jar exactly one tenth of what it initially contains into each of the other nine jars. At the end of these ten operations, each jar will contain one tenth of what is inside each jar initially. Since the total amount of milk remains unchanged, each jar will contain one tenth of the total amount of milk.

Second Solution:

Let k be the number of jars which contain the smallest amount of milk. If $k = 10$, there is nothing to do. Suppose $k < 10$. Let each of these k jars contain m units of milk. There is another jar which contains the next smallest amount of milk, say n units. Pour $\frac{n-m}{10}$ units from the jar containing n units into each of the other nine jars. Then each of the k jars with m units before now has $m + \frac{n-m}{10} = \frac{9m+n}{10}$ units of milk. The jar with n units before now has $n - \frac{9(n-m)}{10} = \frac{9m+n}{10}$ units also. In either case, the value of k has been increased by 1. Performing this procedure at most 9 times, we can raise k to 10.

2. Assign spatial coordinates to the unit cubes, each dimension ranging from 1 to 10. If all cubes are in the same colour orientation, there is nothing to prove. Hence we may assume that (i, j, k) and $(i + 1, j, k)$ do not. Since they share a left-right face, let the common colour be red. We may assign blue to the front-back faces of (i, j, k) . Then its top-bottom faces are white, the front-back faces of $(i + 1, j, k)$ are white and the top-bottom faces of $(i + 1, j, k)$ is blue. Now $(i, j + 1, k)$ share a white face with (i, j, k) while $(i + 1, j + 1, k)$ share a blue face with $(i + 1, j, k)$. Since $(i, j + 1, k)$ and $(i + 1, j + 1, k)$ share a left-right face, the only available colour is red. It follows that the $1 \times 2 \times 10$ block with $(i, 1, k)$ and $(i + 1, 1, k)$ at one end and $(i, 10, k)$ and $(i + 1, 10, k)$ at the other has 1×10 faces left and right which are all red. Similarly, if we carry out the expansion vertically, we obtain a $2 \times 10 \times 10$ block with 10×10 faces left and right which are all red. Finally, if we carry out the expansion sideways, we will have the left and right faces of the large cube all red.
3. Suppose $a = b$. Then $a + a^2 = a(a + 1)$ is a power of 2, so that each of a and $a + 1$ is a power of 2. This is only possible if $a = 1$. Suppose $a \neq b$. By symmetry, we may assume that $a > b$, so that $a^2 + b > a + b^2$. Since their product is a power of 2, each is a power of 2. Let $a^2 + b = 2^r$ and $a + b^2 = 2^s$ with $r > s$. Then $2^s(2^{r-s} - 1) = 2^r - 2^s = a^2 + b - a - b^2 = (a - b)(a + b - 1)$. Now $a - b$ and $a + b - 1$ have opposite parity. Hence one of them is equal to 2^s and the other to $2^{r-s} - 1$. If $a - b = 2^s = a + b^2$, then $-b = b^2$. If $a + b - 1 = 2^s = a + b^2$, then $b - 1 = b^2$. Both are contradictions. Hence there is a unique solution $a = b = 1$.

4. Extend AB to P' so that $BP' = BP = CQ$. Then $BP'CQ$ is a parallelogram so that $P'Q$ and BC bisect each other at a point K . Let AK intersect BD at G' and let QG' intersect AB at R' . Since K is the midpoint of BC , its distance from BD is half the distance of C from BD , which is equal to the distance of A from BD . It follows that $AG' = 2KG'$. Since K is the midpoint of $P'Q$, G' is the centroid of triangle $AP'Q$. Hence $QG' = 2R'G'$ and R' is the midpoint of AP' . Let R be the midpoint of AP and let QR intersect BD at G . Then RR' is parallel to PP' , which is in turn parallel to BD . Hence $QG = 2RG$ so that G is the centroid of triangle APQ .



5. Solution by Central Jury:

- (a) Suppose $n + 1 = k^2$ for some positive integer k . We take the lightest k objects with total weight $1 + 2 + \dots + k = \frac{k(k+1)}{2}$ grams. The average weight of the remaining objects is $\frac{(k+1)+(k^2-1)}{2} = \frac{k(k+1)}{2}$ grams also.
- (b) The total weight of the n objects is $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ grams. Let T grams be the total weight of the k chosen objects. This is also the average weight of the remaining $n - k$ objects. Hence $\frac{n(n+1)}{2} = T(n - k + 1)$. Now

$$2T(n - k + 1) = n(n + 1) > n^2 + n - k^2 + k = (n + k)(n - k + 1),$$

so that $2T > n + k$. If we choose the lightest k objects, then T attains its maximum value $\frac{(k+1)+n}{2}$, so that $2T \leq n + k + 1$. It follows that we must have $2T = n + k + 1$, and we must take the lightest k objects. Then $\frac{n+k+1}{2} = T = 1 + 2 + \dots + k = \frac{k^2+k}{2}$, so that $n + 1 = k^2$.

6. Solution by Olga Ivrii:

Partition the infinite chessboard into $n \times n$ subboards by horizontal and vertical lines n units apart. Within each subboard, assign the coordinates (i, j) to the square at the i -th row and the j -th column, where $1 \leq i, j \leq n$. Whenever an $n \times n$ cardboard is placed on the infinite chessboard, it covers n^2 squares all with different coordinates. The total number of times squares with coordinates $(1,1)$ is covered is 2009. Since 2009 is odd, at least one of the squares with coordinates $(1,1)$ is covered by an odd number of cardboards. The same goes for the other $n^2 - 1$ coordinates. Hence the total number of squares which are covered an odd number of times is at least n^2 .

7. Solution by Central Jury:

We construct a graph, with the vertices representing the islands and the edges representing connecting routes. The graph may have one or more connected components. Since the total number of vertices is odd, there must be a connected component with an odd number of vertices. Olga chooses from this component the largest set of independent edges, that is, edges no two of which have a common endpoint. She will colour these edges red. Since the number of vertices is odd, there is at least one vertex which is not incident with a red vertex. Olga will start the tour there. Suppose Max has a move. It must take the tour to a vertex incident with a red edge. Otherwise, Olga could have colour one more edge red. Olga simply continue the tour by following that red edge. If Max continues to go to vertices incident with red edges, Olga will always have a ready response. Suppose somehow Max manages to get to a vertex not incident with a red edge. Consider the tour so far. Both the starting and the finishing vertices are not incident with red edges. In between, the edges are alternately red and uncoloured. If Olga interchanges the red and uncoloured edges on this tour, she could have obtained a larger independent set of edges. This contradiction shows that Max could never get to a vertex not incident with red edges, so that Olga always wins if she follows the above strategy.