International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2009.**

1. After a gambling session, each of one hundred pirates calculated the amount he had won or lost. Money could only change hands in the following way. Either one pirate pays an equal amount to every other pirate, or one pirate receives the same amount from every other pirate. Each pirate had enough money to make any payment. Prove that after several such steps, it was possible for all the winners to receive exactly what they had won and for all losers to pay exactly what they had lost.

2. A non-square rectangle is cut into N rectangles of various shapes and sizes. Prove that one can always cut each of these rectangles into two rectangles so that one can construct a square and rectangle, each figure consisting of one piece from each of the N rectangles.

3. Every edge of a tetrahedron is tangent to a given sphere. Prove that the three line segments joining the points of tangency of the three pairs of opposite edges of the tetrahedron are concurrent.

4. Denote by $[n]!$ the product of 1, 11, $\ldots$, $11\ldots1$, where the last factor has n ones. Prove that $[n+m]!$ is divisible by $[n]![m]!$.

5. Let XYZ be a triangle. The convex hexagon $ABCDEF$ is such that AB, CD and EF are parallel and equal to XY, YZ and ZX, respectively. Prove that the area of the triangle with vertices at the midpoints of BC, DE and FA is not less than the area of triangle XYZ.

6. Olga and Max visited a certain Archipelago with 2009 islands. Some pairs of islands were connected by boats which run both ways. Olga chose the first island on which they land. Then Max chose the next island which they could visit. Thereafter, the two took turns choosing an accessible island which they had not yet visited. When they arrived at an island which was connected only to islands they had already visited, whoever's turn to choose next would be the loser. Prove that Olga could always win, regardless of the way Max played and regardless of the way the islands were connected.

7. At the entrance to a cave is a rotating round table. On top of the table are n identical barrels, evenly spaced along its circumference. Inside each barrel is a herring either with its head up or its head down. In a move, Ali Baba chooses from 1 to n of the barrels and turns them upside down. Then the table spins around. When it stops, it is impossible to tell which barrels have been turned over. The cave will open if the heads of the herrings in all n barrels are all up or are all down. Determine all values of n for which Ali Baba can open the cave in a finite number of moves.

Note: The problems are worth 4, 6, 7, 9, 9, 12 and 14 points respectively.

Solution to Senior A-Level Fall 2009

1. **First Solution by Wen-Hsien Sun:**
A pirate who owes money is put in group A, and the others are put in group B. Each pirate in group A puts the full amount of money he owes into a pot, and the pot is shared equally among all 100 pirates. For each pirate in group B, each of the 100 pirates puts $\frac{1}{100}$-th of the amount owed to him in a pot, and this pirate takes the pot. We claim that all debts are then settled. Let a be the total amount of money the pirates in group A owe, and let b be the total amount of money owed to the pirates in group B. Clearly, $a = b$. Each pirate in group A pays off his debt, takes back $\frac{a}{100}$ and then pays out another $\frac{b}{100}$. Hence he has paid off his debt exactly. Each pirate in group B takes in $\frac{a}{100}$, pays out $\frac{b}{100}$ and then takes in what is owed him. Hence the debts to him have been settled too.

   Second Solution:
Let M units be the maximum amount of money won by one pirate. Such a pirate brings no money to the changing room. A pirate who wins N units where $N < M$ brings to the changing room an amount of money equal to $M - N$. A pirate who loses N units brings to the changing room an amount of money equal to $M + N$. Let k be the number of pirates who has the smallest amount of money in the changing room. If $k = 100$, there is nothing to do. Suppose $k < 100$. Let each of these k pirates have m units of money. There is another pirate who has the next smallest amount of money, say n units. He gives $\frac{n-m}{100}$ units to each of the other 99 pirates. Then each of the k pirates with m units before now has $m + \frac{n-m}{100} = \frac{99m+n}{10}$ units of money. The pirate with n units before now has $n - \frac{99(n-m)}{100} = \frac{99m+n}{100}$ units also. Thus the value of k has been increased by 1. Performing this procedure at most 99 times, we can raise k to 100. Each of the pirates now has M units of money, meaning that all debts have been collected or paid accordingly.

2. **Solution by Rosu Cristina and Jonathan Zung, independently:**
Let the given rectangle R have length m and width n with $m > n$. Contract the length of R by a factor of $\frac{n}{m}$, resulting in an $n \times n$ square. For each of the N rectangle in R, the corresponding rectangle in S has the same width but shorter length. Thus we can cut the former into a primary piece congruent to the latter, plus a secondary piece. Using S as a model, the N primary pieces may be assembled into an $n \times n$ square while the N secondary pieces may be assembled into an $(m - n) \times n$ rectangle.

3. Let the points of tangency to the sphere of AB, AC, DB and DC be K, L, M and N respectively. The line KL intersects the line BC at some point P not between B and C. By the converse of the undirected version of Menelaus' Theorem, $1 = \frac{BP}{PC} \cdot \frac{CL}{LA} \cdot \frac{AK}{KB} = \frac{BP}{PC} \cdot \frac{CL}{KB}$ since $LA = AK$. Since $CL = CN$, $KB = MB$ and $ND = DM$, $1 = \frac{BP}{PC} \cdot \frac{CN}{MB} = \frac{BP}{PC} \cdot \frac{CN}{ND} \cdot \frac{DM}{MB}$. By the undirected version of Menelaus' Theorem, P, M and N are collinear. It follows that K, L, M and N are coplanar, so that KN intersects LM. Similarly, the line joining the points of tangency to the sphere of AD and BC also intersects KN and LM. Since the three lines are not coplanar, they must intersect one another at a single point.

4. **Solution by Jonathan Zung:**

Define $f(n) = 111\ldots1$ with n 1s and $f(0) = 1$ so that [0]!=1. Define $\begin{bmatrix} n \\ k \end{bmatrix} = \dfrac{[n]!}{[k]![n-k]!}$ for $0 \le k \le n$. We use induction on n to prove that $\begin{bmatrix} n \\ k \end{bmatrix}$ is always a positive integer for all $n \ge 1$. For $n = 0$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \dfrac{[0]!}{[0]![0]!} = 1$. Suppose the result holds for some $n \ge 0$. Consider the next case.

$$\begin{aligned}
\begin{bmatrix} n+1 \\ k \end{bmatrix} &= \frac{[n+1]!}{[k]![n+1-k]!} \\
&= \frac{[n]!f(n+1)}{[k]![n+1-k]!} \\
&= \frac{[n]!f(n+1-k)10^k}{[k]![n-k]!f(n+1-k)} + \frac{[n]!f(k)}{[k-1]![n+1-k]!} \\
&= 10^k \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}.
\end{aligned}$$

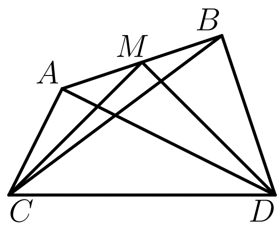
Since both terms in the last line are positive integers, the induction argument is complete. In particular, for any positive integers m and n, $\begin{bmatrix} m+n \\ m \end{bmatrix} = \dfrac{[m+n]!}{[m]![n]!}$ is a positive integer, so that $[m+n]!$ is divisible by $[m]![n]!$.

5. **Solution by Central Jury:**

Denote the area of a polygon P by $[P]$. We first establish a preliminary result.

Lemma.

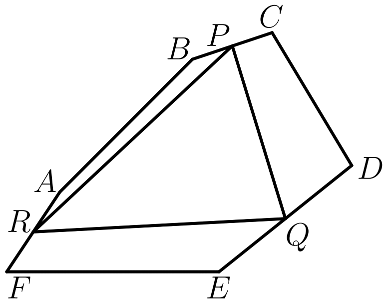
Let M be the midpoint of a segment AB which does not intersect another segment CD. Then $[CMD] = \frac{[CAD]+[CBD]}{2}$.



Proof:

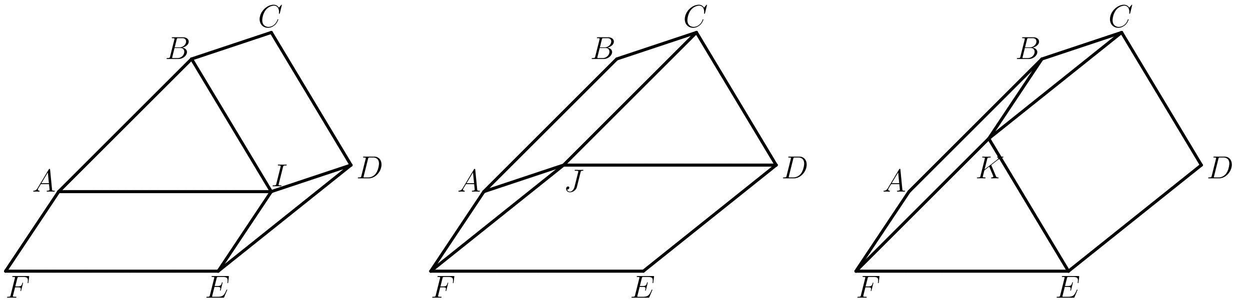
Since M is the midpoint of AB, we have $[CAD] = [ABDC] - [BAD] = [ABDC] - 2[BMD]$ and $[CBD] = [ABDC] - [ABC] = [ABDC] - 2[AMC]$. Hence

$$2[CMD] = 2([ABDC] - [AMC] - [BMD]) = [CAD] + [CBD].$$



Returning to the problem, let P, Q and R be the respective midpoints of BC, DE and FA. By the Lemma, we have

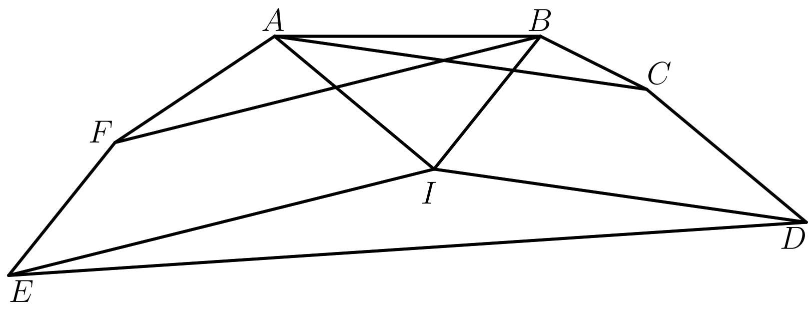
$$\begin{aligned}[PQR] &= \frac{1}{2}([BQR]+[CQR]) \\ &= \frac{1}{4}([BDR]+[BER]+[CDR]+[CER]) \\ &= \frac{1}{8}([BAD]+[BFD]+[BAE]+[BFE]+[CAD]+[CFD]+[CAE]+[CFE]).\end{aligned}$$



Let I be the point such that ABI is congruent to XYZ. Then $BCDI$ and $EFAI$ are parallelograms. Since $ABCDEF$ is convex, I is inside the hexagon. Hence $[XYZ] < [ABCDEF]$. Note that the distance of D from AB is equal to the sum of the distances from C and I to AB, Hence $[BAD] = [BAC] + [BAI] = [BAC] + [XYZ]$. Similarly, $[BAE] = [BAF] + [XYZ]$. Let J and K be the points such that JCD and FKE are congruent to XYZ. Then we have $[ACD] = [BCD] + [XYZ]$, $[FCD] = [ECD] + [XYZ]$, $[BFE] = [AFE] + [XYZ]$ and $[CFE] = [DFE] + [XYZ]$. It follows that $[PQR] = \frac{1}{8}(2[ABCDEF] + 6[XYZ]) > [XYZ]$.

Remark:

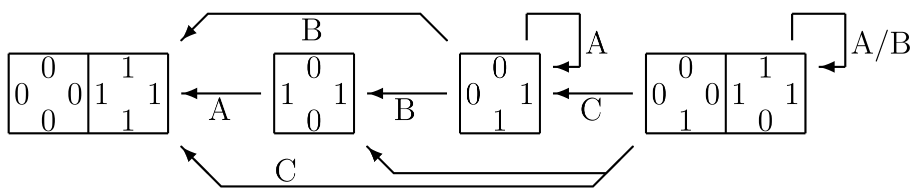
The solution above makes a reasonable assumption that XYZ and $ABCDEF$ are in the same orienttion. If they are not, the first of the three diagrams above may look like the one below, and minor modifications to the argument are necessary. However, this complication is a mere detraction to an already very nice problem.



6. **Solution by Central Jury:**
We construct a graph, with the vertices representing the islands and the edges representing connecting routes. The graph may have one or more connected components. Since the total number of vertices is odd, there must be a connected component with an odd number of vertices. Olga chooses from this component the largest set of independent edges, that is, edges no two of which have a common endpoint. She will colour these edges red. Since the number of vertices is odd, there is at least one vertex which is not incident with a red vertex. Olga will start the tour there. Suppose Max has a move. It must take the tour to a vertex incident with a red edge. Otherwise, Olga could have colour one more edge red. Olga simply continue the tour by following that red edge. If Max continues to go to vertices incident with red edges, Olga will always have a ready response. Suppose somehow Max manages to get to a vertex not incident with a red edge. Consider the tour so far. Both the starting and the finishing vertices are not incident with red edges. In between, the edges are alternately red and uncoloured. If Olga interchanges the red and uncoloured edges on this tour, she could have obtained a larger independent set of edges. This contradiction shows that Max could never get to a vertex not incident with red edges, so that Olga always wins if she follows the above strategy.

7. **Solution by Hsin-Po Wang:**
The task is guaranteed to succeed if and only if n is a power of 2. Suppose n is not a power of 2. Then it has an odd prime factor p. Choose p evenly spaced barrels and make sure that the herrings inside are not all pointing the same way. Ignore all other barrels. At any point, let the herrings in r barrels are pointing up while the herrings in the other s barrels are pointing down. Since $r + s = p$ is odd, $r \neq s$. We may assume that $r > s$. In order for Ali Baba to succeed, he must turn over all r barrels of the first kind or all s barrels of the second kind. A pagan god who is having fun with Ali Baba can spin the table so that if Ali Baba plans to turn over r barrels, the herring in at least one of them is pointing down; and if Ali Baba plans to turn over s barrels, the herring in all of them are pointing up. This way, Ali Baba will never be able to open the cave.

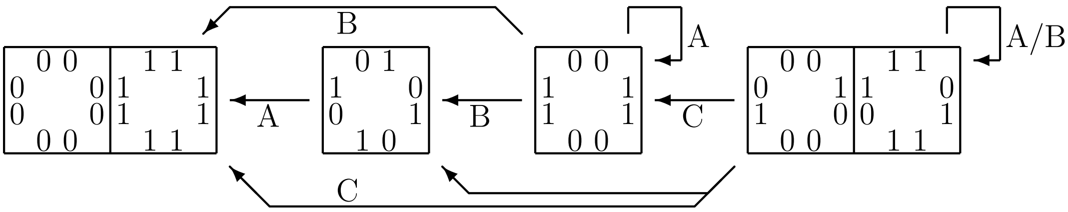
If $n = 2^k$ for some non-negative integer k, we will prove by induction on k that Ali Baba can open the cave. The case $k = 0$ is trivial as the cave opens automatically. The case $k = 1$ is easy. If the cave is not already open, turning one barrel over will do. For $k = 2$, let 0 or 1 indicate whether the herring is heads up or heads down.



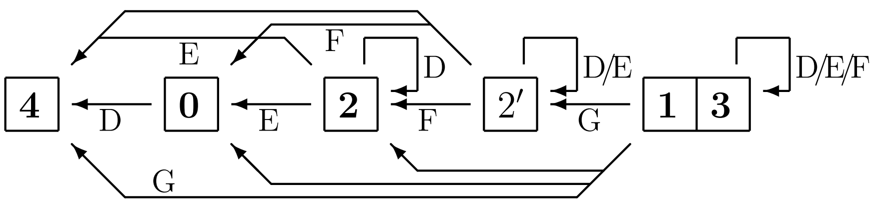
The diagram above represents the four possible states the table may be in, as well as the transition between states by the following operations. Operation **A:** Turn over any two opposite barrels. Operation **B:** Turn over any two adjacent barrels. Operation **C:** Turn over any one barrel. By performing the sequence **ABACABA**, the cave will open. The first state is called an *absorbing* state, in that once there, no further transition takes place as the cave will open immediately.

The second state becomes the first state upon the first operation A. The third state remains in place during the first operation A, but will become either the first state or the second state upon the first operation B. In the latter case, it will become the first state upon the second operation A. The fourth state remains in place during the first three operations, but will become any of the other three states upon the operation C. It will become the first state at the latest after three more operations.

The success of the case $k = 2$ paves the way for the case $k = 3$. The process is typical of the general inductive argument so that we give a detailed analysis. The idea is to treat each pair of diametrically opposite barrels as a single entity.



The above diagram, which is essentially copied from that for $k = 2$, is part of a much bigger state-transition diagram for $k = 3$. Here, all the states have the property that opposite pairs of barrels are all matching, that is, both are 0 or both are 1. The operations are modified from those in the case $k = 2$ as follows.Operation **A:** Turn over every other pair of opposite barrels; in other words, turn over every other barrel. Operation **B:** Turn over any two adjacent pairs of opposite barrels. Operation **C:** Turn over any pair of opposite barrels. By performing the sequence **ABACABA**, the cave will open. These states together form an expanded absorbing state in the overall diagram below.



Here, the box marked m contains all states with m matching opposite pairs, where $0 \leq m \leq 4$. The box marked 4 is the expanded absorbing state mentioned above. The states with 2 matching pairs are classified according to whether these matching are alternating or adjacent. The former states are contained in the box marked 2 while the latter states are contained in the box marked $2'$. We have three new operations. Operation **D:** Turn over any 4 adjacent barrels. Operation **E:** Turn over any 2 barrels separated by one other barrel. Operation **F:** Turn over any two adjacent barrels. Operation **G:** Turn over any barrel.

Let **X** denote the sequence ABACABA. Then the sequence for the case $k = 3$ is

XDXEXDXFXDXEXDXGXDXEXDXFXDXEXDX.

We keep repeating X to clear any state that has entered the box marked 4, to prevent them from returning to another box. Whatever the state the table is in, the cave will open by the end of this sequence.

The general procedure is now clear. We treat each opposite pair as a single entity, thereby reducing to the preceding case. Then we moving progressively all states into the expanded absorbing state. Thus the task is possible whenever n is a power of 2.