International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2009.**

1. Is it possible to cut a square into nine squares and colour one of them white, three of them grey and five of them black, such that squares of the same colour have the same size and squares of different colours will have different sizes?

2. There are forty weights: 1, 2, ..., 40 grams. Ten weights with even masses were put on the left pan of a balance. Ten weights with odd masses were put on the right pan of the balance. The left and the right pans are balanced. Prove that one pan contains two weights whose masses differ by exactly 20 grams.

3. A cardboard circular disk of radius 5 centimetres is placed on the table. While it is possible, Peter puts cardboard squares with side 5 centimetres outside the disk so that:
(1) one vertex of each square lies on the boundary of the disk;
(2) the squares do not overlap;
(3) each square has a common vertex with the preceding one.
Find how many squares Peter can put on the table, and prove that the first and the last of them must also have a common vertex.

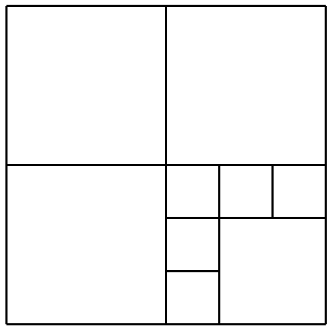
4. A 7-digit passcode is called good if all digits are different. A safe has a good passcode, and it opens if seven digits are entered and one of the digits matches the corresponding digit of the passcode. Is there a method of opening the safe box with an unknown passcode using less than 7 attempts?

5. A new website registered 2000 people. Each of them invited 1000 other registered people to be their friends. Two people are considered to be friends if and only if they have invited each other. What is the minimum number of pairs of friends on this website?

Note: The problems are worth 3, 4, 4, 5 and 5 points respectively.

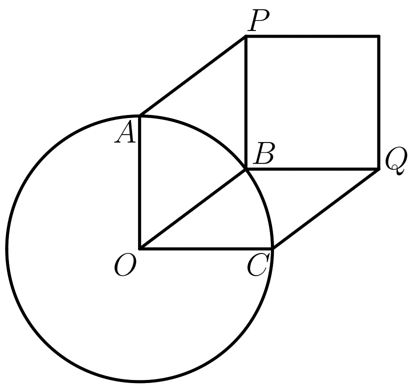
Solution to Junior O-Level Fall 2009

1. The diagram below shows that a 6×6 square can be cut into one 2×2 square, three 3×3 squares and five 1×1 squares.



2. Suppose to the contrary that no two weights in the same pan differ in mass by exactly 20 grams. Then in the right pan, we must have put in exactly one weight from each of the following ten pairs: (1,21), (3,23), ..., (19,39). The total mass in the right pan is $1 + 3 + \cdots + 19 + 20k = 100 + 20k$, where k is the number of times we chose the heavier weight from a pair. This is a multiple of 4. Similarly, the total mass in the left pan is $2 + 4 + \cdots + 20 + 20h = 110 + 20h$, where h is the number of times we chose the heavier weight from a pair. This is not a multiple of 4. We have a contradiction as the two pans cannot possibly balance.

3. Let O be the centre of the circle, A, B and C be the points of contact with the circle of three squares in order, and P and Q be the common vertices of these squares. Call OA, OB and OC the root canals of the respective squares. Then $OAPB$ and $OBQC$ are rhombi. Moreover, $\angle PBQ = 90°$. Hence $\angle AOC = 90°$. This means that every two alternate root canals are perpendicular. It follows that there must be 8 root canals, and the last square must have a common vertex with the first.



4. In six attempts, try entering 0123456, 0234561, 0345612, 0456123, 0561234 and 0612345. Since the correct passcode uses 7 different digits, it must use at least 3 of the digits 1, 2, 3, 4, 5 and 6. At most one of these 3 can be in the first place. The other 2 must match one of our attempts.

5. Pretend that the 2000 people are seated at a round table, evenly spaced. Each invites the next 1000 people in clockwise order. Then only two people who are diametrically opposite to each other become friends. This shows that the number of pairs of friends may be as low as 1000. On the other hand, each person sends out 1000 invitations and receives 1000 invitations, and must become friends with at least one other person. It follows that the minimum number of pairs of friends is 1000.