International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2014.**

1. Half of all entries in a square table are plus signs, and the remaining half are minus signs. Prove that either two rows or two columns contain the same number of plus signs.

2. Prove that any polygon with an incircle has three sides that can form a triangle.

3. Is it possible to divide all positive divisors of 100!, including 1 and 100!, into two groups of equal size such that the product of the numbers in each group is the same?

4. On a circular road there are 25 equally spaced booths, each with a patrolman numbered from 1 to 25 in some order. The patrolmen switch booths by moving along the road, so that their numbers are from 1 to 25 in clockwise order. If the total distance travelled by the patrolmen is as low as possible, prove that one of them remains in the same booth.

5. In triangle ABC, $\angle A = 90°$. Two equal circles tangent to each other are such that one is tangent to BC at M and to AB, and the other is tangent to BC at N and to CA. Prove that the midpoint of MN lies on the bisector of $\angle A$.

6. A *uniform* number is a positive integer in which all digits are the same. Prove that any n-digit positive integer can be expressed as the sum of at most $n+1$ uniform numbers.
   Example: The numbers 4, 111 and 999999 are uniform.

7. A spiderweb is a square grid with 100×100 nodes, at 100 of which flies are stuck. Starting from a corner node of the web, a spider crawls from a node to an adjacent node in each move. A fly stuck at the node where the spider is wi ll be eaten. Can the spider always eat all the flies in no more than

   (a) 2100 moves;

   (b) 2000 moves?

Note: The problems are worth 4, 5, 6, 7, 8, 8 and 5+5 points respectively.

Solution to Junior A-Level Fall 2014

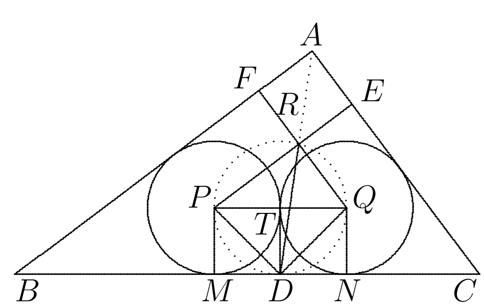
1. Since half of the entries are plus signs, the table must be $2n \times 2n$ for some positive integer n. Suppose the numbers of plus signs in the $2n$ rows are all different. Then they must be 0, 1, 2, ..., $2n$, with one omitted. Since the total number of plus signs is $2n^2$, the one omitted is n. Then there is a row with $2n$ plus signs and another row with 0 signs. This means that we do not have a column with $2n$ plus signs, and we do not have a column with 0 plus signs. Hence the numbers of plus signs in the $2n$ columns cannot be all different.

2. Let BC be the longest side of the polygon and let AB and CD be the sides on either side of BC. Then $AB+BC > BC \geq CD$ and $BC+CD > BC \geq AB$. Let the incircle be tangent to AB, BC and CD at K, L and M respectively. Then $AB+CD > KB+CM = BL+CL+BC$. Hence these three sides of the polygon can form a triangle.

3. In the prime factorization of 100!, each of the primes 97 and 89 appears only once. Hence the number of positive divisors of 100! is divisible by $(1+1)(1+1) = 4$. They form an even number of pairs whose product is 100!. If we put half of the pairs in one group and the remaining pairs in the other group, then the two groups are of equal size and the product of the numbers in each group is the same.

4. **Solution by Po-Sheng Wu:**
   Consider the motion plan which accomplishes the desired result in which the total distance covered by the patrolmen is minimum. Suppose all of them move. Let m be the number of those who move clockwise and n be the number of those who move counterclockwise. Then $m \neq n$ since $m+n = 25$. By symmetry, we may assume that $m < n$. Ask each patrolman who moves clockwise to go one booth farther and each patrolmen who moves counterclockwise to stop one booth earlier. Then the patrolmen's numbers will still be in order. However, the total distance they have covered will be reduced by $n - m$ times the distance between two booths. This contradicts the minimality assumption on the original motion plan.

5. **Solution by Po-Sheng Wu:**
   Let P be the centre circle tangent to BC at M and Q be the centre of the circle tangent to BC at N. Let T be the point of tangency of these two circles and let D be the midpoint of MN. Drop perpendiculars from P to CA at E and from Q to AB at F, intersecting each other at R. Then $AERF$ is a square whose side length is equal to the common radii of the two circles. Hence $\angle RAF = \angle ARF = 45°$. Now $DMPT$ and $DNQT$ are also squares. Hence $\angle PDQ = 90° = \angle QRP$. Hence $DPQR$ is a cyclic quadrilateral and $\angle DRQ = \frac{1}{2}\angle DTQ = 45°$. It follows that A, R and D are collinear, so that D lies on the bisector of $\angle A$.



6. **Solution by Po-Sheng Wu:**
   Denote by d_n the n-digit number in which every digit is d. We prove by induction on n that every positive integer less than or equal to 1_n is the sum of at most n uniform numbers. The result is trivial for $n = 1$. Suppose it holds for some $n \geq 1$. Consider a number $m \leq 1_{n+1}$. Subtract from m the largest uniform number $u \leq m$. If $m = 1_{n+1}$, then $u = m$ and $m - u = 0$. If $9_n \leq m < 1_{n+1}$, then $u = 9_n$ and $m - u \leq 1_n$. If $d_n \leq m < (d+1)_n$ for some d, $0 \leq d \leq 8$, then $u = d_n$ and $m - u \leq 1_n$. In all cases, $m - u \leq 1_n$ and is a sum of at most n uniform numbers by the induction hypothesis. It follows that m is the sum of at most $m + 1$ uniform numbers. Since an n-digit number is less than 1_{n+1}, it is also the sum of at most $n+1$ uniform numbers.

7. **Solution by Po-Sheng Wu:**
   The answers to both parts are affirmative. We may assume that the spider starts from the top left corner. Partition the spiderweb into ten 100×10 vertical strips, which it will comb through one by one from left to right. All vertical moves are along either the first or the last column of a strip, downwards on odd-numbered strips and upwards on even-numbered strips. The total number of vertical moves is $10 \times 99 = 990$. All horizontal moves are within the same strip back and forth betwen the first and the last columns, at the horizontal levels which contains at least one fly. When the spider reaches the bottom row in an odd-numbered strip or the top row in an even-numbered strip, it moves over to the next strip. The number of horizontal moves used for gobbling up flies is at most $9 \times 100 = 900$ since there are 100 flies. The number of horizontal moves used for changing strips is 9. The number of horizontal moves used for getting to the correct column for chaning strips is at most $9 \times 10 = 90$. It follows that the total number of moves required is at most 990+900+9+90 =1989.