International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2014.**

1. There are 99 sticks of lengths 1, 2, 3, ..., 99. Is it possible to use all of them to form the perimeter of a rectangle?

2. Do there exist ten pairwise distinct positive integers such that their average divided by their greatest common divisor is equal to

   (a) 6;

   (b) 5?

3. K and L are points on the sides AB and BC of a square $ABCD$ respectively, such that $KB = LC$. P be the point of intersection of AL and CK. Prove that DP and KL are perpendicular.

4. In the 40 tests Andrew had taken, he got 10 As, 10 Bs, 10 Cs and 10 Ds. A score is said to be *unexpected* if this particular score has appeared up to now fewer times than any of the other three scores. Without knowing the order of these 40 scores, is it possible to determine the number of unexpected ones?

5. There are $n > 1$ right triangles. In each triangle, Adam chooses a leg and calculates the sum of their lengths. Then he calculates the sum of the lengths of the remaining legs. Finally, he calculates the sum of the lengths of the hypotenuses. If these three numbers are the side lengths of a right triangle, prove that the n triangles are similar to one another for

   (a) $n = 2$;

   (b) an arbitrary positive integer n.

Note: The problems are worth 3, 2+2, 5, 5 and 2+3 points respectively.

Solution to Junior O-Level Fall 2014

1. We first use the sticks of lengths 1, 2, 3, 4, 5, 6 and 7 to form the perimeter of a square, with side length 7=6+1=5+2=4+3. Divide the remaining 92 sticks into groups of four: (8,9,10,11), (12,13,14,15), ..., (96,97,98,99). For each group, add the longest and the shortest sticks to the top edge of the rectangle and the other two sticks to the bottom edge.

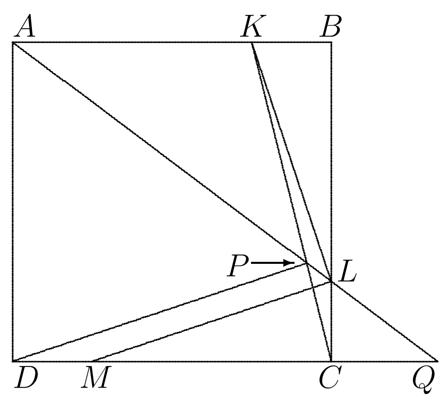
2. (a) We may take the ten numbers to be 1, 2, 3, 4, 5, 6, 7, 8, 9 and 15. Their sum is 60 so that their average is 6. Their greatest common divisor is 1, and indeed $6 \div 1 = 6$.

   (b) This is impossible. The greatest common divisor appears everywhere and can be cancelled out. Hence we may take it to be 1. However, the smallest ten distinct positive integers are 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10, with an average of 5.5. Thus the quotient can never be equal to 5.

3. We may take $KB = LC = 1$ and $AK = BL = \lambda$ so that $AB = BC = \lambda + 1$. Let M be the point on CD such that $MD = 1$, and let Q be the point of intersection of the extensions of AL and DC. Since triangles ABL and QCL are similar, $\frac{AL}{QL} = \frac{AB}{QC} = \frac{BL}{CL} = \lambda$. Triangles AKP and QCP are also similar, so that $\frac{AP}{QP} = \frac{AK}{QC} = \frac{\lambda^2}{\lambda+1}$. Hence

$$\begin{aligned} PL &= QP - QL \\ &= \frac{\lambda+1}{\lambda^2}AP - \frac{1}{\lambda}AL \\ &= \left(\frac{1}{\lambda} + \frac{1}{\lambda^2}\right)(AL - PL) - \frac{1}{\lambda}AL \\ &= \frac{1}{\lambda}QL - \frac{\lambda+1}{\lambda^2}PL. \end{aligned}$$

   It follows that $\frac{PL}{QL} = \frac{\lambda^2+\lambda+1}{\lambda}$. On the other hand, $\frac{MQ}{MD} = MC + CQ = \frac{\lambda^2+\lambda+1}{\lambda}$ also. Hence DP is parallel to ML. Since triangles KBL and LCM are congruent, KL is perpendicular to LM. The desired conclusion follows.



4. Consider the first A, the first B, the first C and the first D that Andrew gets. The last one to come along must be unexpected, and none of the other three can be unexpected. The same applies to the second A, the second B, the second C and the second D that he gets, and so on. It follows that exactly 10 of the scores are unexpected.

5. (a) For $i = 1$ or 2, let the side lengths of the i-th triangle be a_i, b_i and c_i with $a_i^2 + b_i^2 = c_i^2$. From $(a_1 + a_2)^2 + (b_1 + b_2)^2 = (c_1 + c_2)^2$, we have $a_1a_2 + b_1b_2 = c_1c_2$. It follows that $(a_1a_2 + b_1b_2)^2 = (a_1^2 + b_1^2)(a_2^2 + b_2^2)$. This may be rewritten as $(a_1b_2 - b_1a_2)^2 = 0$. Hence $a_1b_2 = b_1a_2$, which is equivalent to $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. Since both triangles are right triangles, they are similar to each other.

   (b) For $1 \le i \le n$, let the side lengths of the i-th triangle be a_i, b_i and c_i with $a_i^2 + b_i^2 = c_i^2$. We have $(a_1 + a_2 + \cdots + a_n)^2 + (b_1 + b_2 + \cdots + b_n)^2 = (c_1 + c_2 + \cdots + c_n)^2$. After expansion and cancellation, we obtain, for $1 \le i < j \le n$, $a_ia_j + b_ib_j = c_ic_j$. As in (a), we have $\frac{a_i}{b_i} = \frac{a_j}{b_j}$. Since all triangles are right triangles, they are similar to one another.