**International Mathematics**

**TOURNAMENT OF THE TOWNS**

**Senior A-Level Paper** **Fall 2014.**

1. Prove that any polygon with an incircle has three sides that can form a triangle.

2. On a circular road there are 25 equally spaced booths, each with a patrolman numbered from 1 to 25 in some order. The patrolmen switch booths by moving along the road, so that their numbers are from 1 to 25 in clockwise order. If the total distance travelled by the patrolmen is as low as possible, prove that one of them remains in the same booth.

3. Gregory writes down 100 numbers on a blackboard and calculates their product. In each move, he increases each number by 1 and calculates their product. What is the maximum number of moves Gregory can make if the product after each move does not change?

4. The incircle of triangle $ABC$ is tangent to $BC$, $CA$ and $AB$ at $D$, $E$ and $F$ respectively. It is given that $AD$, $BE$ and $CF$ are concurrent at a point $G$, and that the circumcircles of triangles $GDE$, $GEF$ and $GFD$ intersect the sides of $ABC$ at six distinct points other than $D$, $E$ and $F$. Prove that these six points are concyclic.

5. Peter prepares a list of all possible words consisting of $m$ letters each of which is T, O, W or N, such that the numbers of Ts and Os is the same in each word. Betty prepares a list of words consisting of $2m$ letters each of which is $T$ or $O$, such that the numbers of Ts and Os is the same in each word. Whose list contains more words?

6. Let $PQR$ be a given triangle. $AFBDCE$ is a non-convex hexagon such that the interior angles at $D$, $E$ and $F$ all have measure $181°$. Moreover, $BD + DC = QR$, $CE + EA = RP$, $AF + FB = PQ$, $\angle EAF = \angle RPQ - 1°$, $\angle FBD = \angle PQR - 1°$ and $\angle DCE = \angle QRP - 1°$. Prove that $\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB}$.

7. Each day, positive integers $m$ and $n$ are chosen by the government. On that day, $x$ grams of gold may be exchanged for $y$ grams of platinum such that $mx = ny$. Initially, $m = n = 1001$. On each subsequent day, the government reduces exactly one of $m$ and $n$ by 1, and after 2000 days, both numbers are equal to 1. Someone has 1 kilogram of each of gold and platinum initially. Without knowing in advance which of $m$ and $n$ will be reduced the next day, can this person have a sure way of performing some clever exchanges, and end up with at least 2 kilograms of each of gold and platinum after these 2000 days?

**Note:** The problems are worth 4, 6, 6, 7, 7, 8 and 10 points respectively.

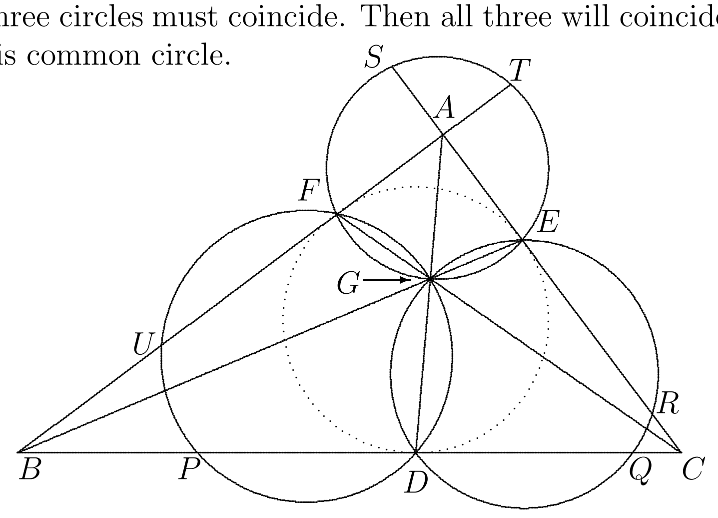
**Solution to Senior A-Level Fall 2014**

1. Let $BC$ be the longest side of the polygon and let $AB$ and $CD$ be the sides on either side of $BC$. Then $AB+BC > BC \geq CD$ and $BC+CD > BC \geq AB$. Let the incircle be tangent to $AB$, $BC$ and $CD$ at $K$, $L$ and $M$ respectively. Then $AB+CD > KB+CM = BL+CL+BC$. Hence these three sides of the polygon can form a triangle.

2. **Solution by Po-Sheng Wu:**
Consider the motion plan which accomplishes the desired result in which the total distance covered by the patrolmen is minimum. Suppose all of them move. Let $m$ be the number of those who move clockwise and $n$ be the number of those who move counterclockwise. Then $m \neq n$ since $m+n = 25$. By symmetry, we may assume that $m < n$. Ask each patrolman who moves clockwise to go one booth farther and each patrolmen who moves counterclockwise to stop one booth earlier. Then the patrolmen's numbers will still be in order. However, the total distance they have covered will be reduced by $n - m$ times the distance between two booths. This contradicts the minimality assumption on the original motion plan.

3. **Solution by Po-Sheng Wu:**
Suppose $a_1, a_2, \ldots, a_{100}$ are real numbers for which there exists a real number $k$ such that $(k + a_1)(k + a_2)\cdots(k + a_{100}) = a_1 a_2 \cdots a_{100}$. Treating this as an equation for $k$, there are at most 100 real roots, one of which is 0. It follows that Gregory can make at most 99 moves. This maximum can be attained if Gregory starts with the numbers from $-99$ to 0. The initial product is 0, and this value is maintained for the next 99 moves, until he hits 100! on the 100th move.

4. **Solution by Po-Sheng Wu:**
Let the circumcircle of triangle $FGD$ intersect the line $AB$ at $U$ and the line $BC$ at $P$. Let the circumcircle of triangle $DGE$ intersect the line $BC$ at $Q$ and the line $CA$ at $R$. Let the circumcircle of triangle $EGF$ intersect the line $CA$ at $S$ and the line $AB$ at $T$. Since $PDFU$ is cyclic, $BP \cdot BD = BU \cdot BF$. Since $BD = BF$, we have $BP = BU$. Since $QEDR$ is cyclic, $BQ \cdot BD = BG \cdot BE$. Since $TFGE$ is cyclic, $BG \cdot BE = BF \cdot BT$. From $BD = BF$, we have $BQ = BT$. It follows that $PQTU$ has a circumcircle. Similarly, we can show that $PRQU$ and $PSTU$ have circumcircles too. The three circles cannot be distinct as otherwise $PQ$, $RS$ and $TU$ are their pairwise radical axes and will be concurrent at the radical centre of the three circles. However, these three lines are the sides of $ABC$, and cannot be concurrent. It follows that two of these three circles must coincide. Then all three will coincide, and $P$, $Q$. $R$, $S$, $T$ and $U$ all lie on this common circle.

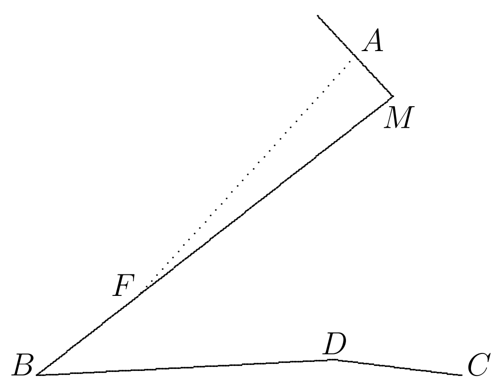


5. **Solution by Po-Sheng Wu:**
For each word Peter writes down, convert every T to TT, every O to OO, every W to OT and every N to TO. Since Peter's word has an equal number of Ts and Os, the new word has an equal number of TTs and OOs. Regardless of the numbers of OTs and TOs, the new word has an equal number of Ts and Os overall, and is therefore on Betty's list. By reversing this conversion process, every word Betty's writes down is on Peter's list. It follows that the two lists have equal length.

6. **Solution by Po-Sheng Wu:**
Let $BD+DC=QR$, $CE+EA=RP$ and $AF+FB=PQ$. We claim that $\frac{BD}{DC}=\frac{CE}{EA}=\frac{AF}{FB}$ if and only if $\angle EAF=\angle RPQ-1°$, $\angle FBD=\angle PQR-1°$ and $\angle DCE=\angle QRP-1°$. We first assume that $\frac{BD}{DC}=\frac{CE}{EA}=\frac{AF}{FB}$. Then triangles $BDC$, $CEA$ and $AFB$ are similar to one another. Hence $\frac{BD+DC}{BC}=\frac{CE+EA}{CA}=\frac{AF+FB}{AB}$. It follows that $\frac{QR}{BC}=\frac{RP}{CA}=\frac{PQ}{AB}$, so that triangles $ABC$ and $PQR$ are similar to each other. Then

$$\angle EAF=\angle CAB-\angle CAE-\angle ECA=\angle RPQ-\angle CAE-\angle ECA=\angle RPQ-1°.$$

Similarly, $\angle FBD=\angle PQR-1°$ and $\angle DCE=\angle QRP-1°$. We now prove the converse, assuming that $\angle EAF=\angle RPQ-1°$, $\angle FBD=\angle PQR-1°$ and $\angle DCE=\angle QRP-1°$. Fix the points $B$, $C$ and $D$. Let $M$ be the fixed point on the same side of $BC$ as $D$ such that $MB=PQ$ and $\angle MBD=\angle PQR-1°$. Then $F$ lies on the segment $MB$ and we have $FM=BM-BF=PQ-BF=AF$. Moreover, $\angle AFM=180°-\angle AFB=1°$. Hence $\angle FMA=\frac{1}{2}(180-1)°=89\frac{1}{2}°$. It follows that $A$ lies on a fixed line through $M$. Let the fixed point $N$ be be defined in an analogous way, using $E$ instead of $F$ and interchanging $B$ and $C$. Then $A$ also lies on a fixed line through $N$. Hence there is at most one possible position for $A$, and $A$ can only exist if we indeed have $\frac{BD}{DC}=\frac{CE}{EA}=\frac{AF}{FB}$. This result holds if 181° amd 1° are replaced respectively by $180°+\theta$ and $\theta$, as long as $AFBDCE$ is still a non-convex hexagon. The diagram below illustrates the case $\theta=10°$, which is much easier to see than if $\theta=1°$.



7. **Solution by Po-Sheng Wu:**

On a day with given $m$ and $n$, the virtual value of $x$ grams of gold and $y$ grams of platinum is taken to be $mx+ny$. This is well-defined since it remains constant regardless of any exchange between the precious metals within the day. Consider an amount of precious metal with virtual value 1 on this day. After $m+n-2$ days, both $m$ and $n$ are 1. Without knowing in advance the order in which $m$ and $n$ are reduced during this period, let $f(m,n)$ be the maximum virtual value which can be guaranteed by performing some clever exchanges. We prove by induction on $m+n$ that $f(m,n)=\frac{m+n-1}{mn}$. Note that $f(1,1)=1$ since no exchange matters, and indeed $\frac{1+1-1}{1^2}=1$. For any $m>1$, we have $f(m,1)=1$. This is because only $m$ can be reduced, and the virtual value is maximized by converting all platinum into gold. Indeed, $\frac{m+1-1}{m}=1$. Similarly, $f(1,n)=1$ for any $n>1$. Suppose $m>1$ and $n>1$. Exchange the precious metals to end up with $\frac{\lambda}{m}$ grams of gold and $\frac{1-\lambda}{n}$ grams of platinum, where $\lambda$ is some parameter to be determined. Note that we indeed have $m(\frac{\lambda}{m})+n(\frac{1-\lambda}{n})=1$. On the following day, there are two possible scenarios. If $m$ is reduced by 1, the new virtual value is $(m-1)(\frac{\lambda}{m})+n(\frac{1-\lambda}{n})=1-\frac{\lambda}{m}$. Then $f(m,n)=f(m-1,n)(1-\frac{\lambda}{m})$. However, if $n$ is reduced by 1, the new virtual value is $m(\frac{\lambda}{m})+(n-1)(\frac{1-\lambda}{n})=1-\frac{1-\lambda}{n}$. Then $f(m,n)=f(m,n-1)(1-\frac{1-\lambda}{n})$. Since we do not know which scenario will take place, we choose the parameter $\lambda$ so that $f(m-1,n)(1-\frac{\lambda}{m})=f(m,n-1)(1-\frac{1-\lambda}{n})$. By the induction hypothesis,

$$\frac{m+n-2}{(m-1)n}\left(\frac{m-\lambda}{m}\right)=\frac{m+n-2}{m(n-1)}\left(\frac{n-1+\lambda}{n}\right).$$

This simplifies to $\frac{m-\lambda}{m-1}=\frac{n-1+\lambda}{n-1}$, which yields $\lambda=\frac{n-1}{m+n-2}$. It follows that

$$\begin{aligned}
f(m,n) &= \frac{m+n-2}{(m-1)n}\left(\frac{m-\frac{n-1}{m+n-2}}{m}\right)\\
&= \frac{m^2+mn-2m-n+1}{mn(m-1}\\
&= \frac{(m-1)^2-n(m-1)}{mn(m-1)}\\
&= \frac{m+n-1}{mn}.
\end{aligned}$$

This completes the inductive argument. Initially, $m=n=1001$ and $x=y=1000$, so that the virtual value is 2002000. Hence the final virtual value which can be guaranteed is $2002000f(1001,1001)=2002000\times\frac{2001}{1001^2}=\frac{4002000}{1001}<4000$. Since we have $m=n=1$ now, $x+y<4000$ so that it is impossible to obtain 4 kilograms of precious metals in any combination.