International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2014.**

1. Do there exist ten pairwise distinct positive integers such that their average divided by their greatest common divisor is equal to

   (a) 6;

   (b) 5?

2. The vertices of triangle ABC are in clockwise order. The triangle is rotated to $A_1B_1C_1$ about $A = A_1$ clockwise through an angle equal to $\angle A$. Then it is rotated to $A_2B_2C_2$ about $B_1 = B_2$ clockwise through an angle equal to $\angle B$. Then it is rotated to $A_3B_3C_3$ about $C_2 = C_3$ clockwise through an angle equal to $\angle C$. This is continued for another three rotations in the same manner, clockwise though angles equal respectively to $\angle A$, $\angle B$ and $\angle C$, until the triangle becomes $A_6B_6C_6$. Prove that $A_6B_6C_6$ coincides with ABC.

3. Peter writes down the sum of every subset of size 7 of a set of 15 distinct integers, and Betty writes down the sum of every subset of size 8 of the same set. If they arrange their numbers in non-decreasing order, can the two lists turn out to be identical?

4. There are $n > 1$ right triangles. In each triangle, Adam chooses a leg and calculates the sum of their lengths. Then he calculates the sum of the lengths of the remaining legs. Finally, he calculates the sum of the lengths of the hypotenuses. If these three numbers are the side lengths of a right triangle, prove that the n triangles are similar to one another for an arbitrary positive integer n.

5. At the beginning, there are some silver coins on a table. In each move, we can either add a gold coin and record the number of silver coins on a blackboard, or remove a silver coin and record the number of golden coins on a whiteboard. At the end, only gold coins remain on the table. Prove that the sum of the numbers on the blackboard is equal to the sum of the numbers on the whiteboard.

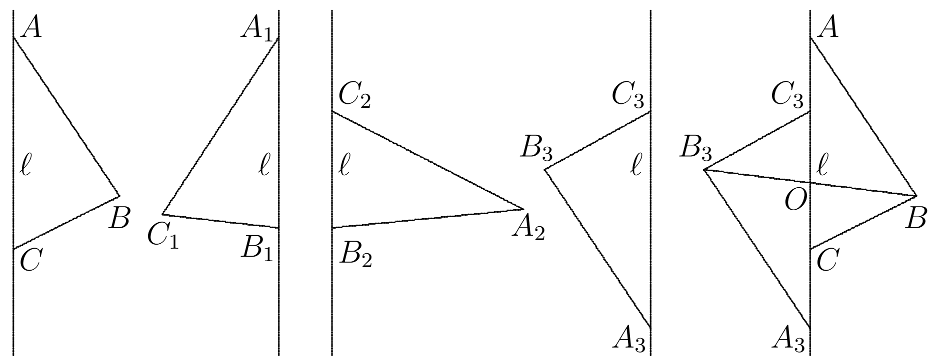
Note: The problems are worth 1+2, 4, 5, 5 and 5 points respectively.

Solution to Senior O-Level Fall 2014

1. (a) We may take the ten numbers to be 1, 2, 3, 4, 5, 6, 7, 8, 9 and 15. Their sum is 60 so that their average is 6. Their greatest common divisor is 1, and indeed $6 \div 1 = 6$.

   (b) This is impossible. The greatest common divisor appears everywhere and can be cancelled out. Hence we may take it to be 1. However, the smallest ten distinct positive integers are 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10, with an average of 5.5. Thus the quotient can never be equal to 5.

2. Let ℓ be the line AC and let B be to its right. In the first rotation, B_1 is on ℓ and $C-1$ is to its left. In the second rotation, C_2 is on ℓ and A_2 is to its right. In the third rotation, A_3 is on ℓ and B_3 is to its left. Since $AC = A_3C_3$, AA_3 and CC_3 have a common midpoint O, which must also be the midpoint of BB_3. Hence $A_3B_3C_3$ may be obtained directly from ABC by a half-turn about O. By symmetry, the next three rotations combine into another half-turn about O, which brings $A_3B_3C_3$ back to ABC.



3. This is possible. Take the set $S = \{-7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$. Then $f(S) = 0$, where f denotes the sum of all the elements in the set. Let $A = \{a_1, a_2, \ldots, a_7\}$ be a subset of S. Let $B = \{b_1, b_2, \ldots, b_7\}$ be such that $b_k = -a_k$ for $1 \leq k \leq 7$. Then $f(B) = -f(A)$. Now $f(S-A) = f(S) - f(A) = -f(A) = f(B)$. Similarly, $f(S-B0 = f(A)$. If $A \neq B$, then the two sums $f(A)$ and $F(b)$ on Peter's list correspond respectively to the two sums $f(S-b)$ and $f(S-A)$ on Betty's list. If $A = B$, then we must have $f(A) = 0$ so that $f(S-A) = 0$ also. In this case, the sum $f(A)$ on Peter's list corresponds to the sum $F(S-A)$ on Betty's list. It follows that the two lists are identical.

4. For $1 \leq i \leq n$, let the side lengths of the i-th triangle be a_i, b_i and c_i with $a_i^2 + b_i^2 = c_i^2$. We have $(a_1+a_2+\cdots+a_n)^2 + (b_1+b_2+\cdots+b_n)^2 = (c_1+c_2+\cdots+c_n)^2$. After expansion and cancellation, we obtain, for $1 \leq i < j \leq n$, $a_ia_j + b_ib_j = c_ic_j$. It follows that $(a_ia_j+b_ib_j)^2 = (a_i^2+b_i^2)(a_j^2+b_j^2)$. This may be rewritten as $(a_ib_j - b_ia_j)^2 = 0$. Hence $a_ib_j = b_ia_j$, which is equivalent to $\frac{a_i}{b_i} = \frac{a_j}{b_j}$. Since all triangles are right triangles, they are similar to one another.

5. Suppose there are m silver coins at the beginning and n gold coins at the end. In each of the $m+n$ moves, we either remove a silver coin or add a gold coin. Place the $m+n$ coins in a row so that, for $1 \leq k \leq m+n$, the coin involved in the k-th move is in the k-th position from the left. Replace each silver coin by a girl facing left, and each gold coin by a boy facing right. Each girl counts the number of boys in front of her, and each boy counts the number of girls in front of him. For any girl and any boy, either both count the other or neither does. Hence the total count by the girls must be equal to the total count by the boys. The numbers counted by the girls are precisely those recorded on the blackboard and the numbers counted by the boys are precisely those recorded on the whiteboard. Hence the sum of the numbers on the blackboard is equal to the sum of the numbers on the whiteboard.