International Mathematics

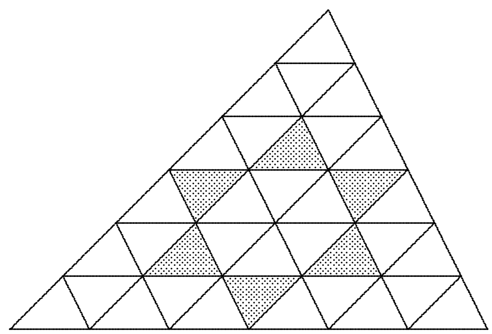
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2016.**

1. Each of ten boys has 100 pokemon cards. In each move, one of the boys gives one card to each of the other boys. What is the minimal number of moves before each boy has a different number of cards?

2. There are 64 positive integers in the squares of an 8×8 board. Whenever the board is covered by 32 dominoes, the sum of the two integers covered by each domino is unique. Is it possible that the largest integer on the board does not exceed 32?

3. The diagram shows an arbitrary triangle dissected into congruent triangles by lines parallel to its sides. Prove that the orthocentres of the six shaded triangles are concyclic.



4. In a 7×7 box, each of the 49 pieces is either dark chocolate or white chocolate. In each move, Alex eats two adjacent pieces along a row, a column or a diagonal, provided that they are of the same kind. What is the mminimum number of chocolates Alex can guarantee to eat, regardless of initial arrangement the pieces?

5. The three pairwise sums of three numbers are recorded on a piece of paper. They are distinct and positive. The three pairwise products of the same three numbers are recorded on another piece of paper. They are also distinct and positive. Later, it is forgotten which piece is which. Is it still possible to determine which piece is which?

6. Let $A_1A_2 \ldots A_{2n}$, $n \geq 5$, be a regular $2n$-gon with the centre O. Diagonals A_2A_{n-1} and A_3A_n intersect at F while diagonals A_1A_3 and A_2A_{2n-2} intersect at E. Prove that $EF = EO$.

7. An examination consists of 20 questions with k multiple choices. For any 10 questions and any of the k^{10} combination of answers to them, some student has given precisely these answers to these questions. Must there exist two students who has given different answers to all 20 questions, where

   (a) $k = 2$;

   (b) $k = 12$?

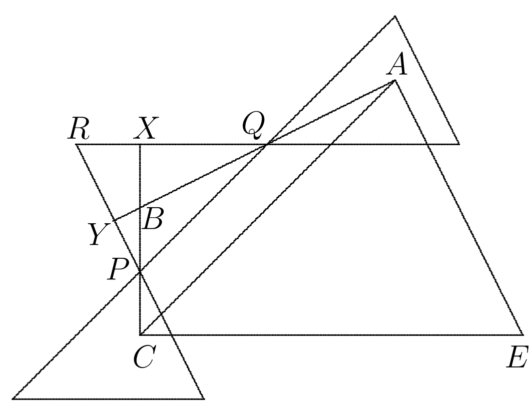
Note: The problems are worth 5, 5, 6, 8, 8, 9 and 5+6 points respectively.

Solution to Junior A-Level Fall 2016

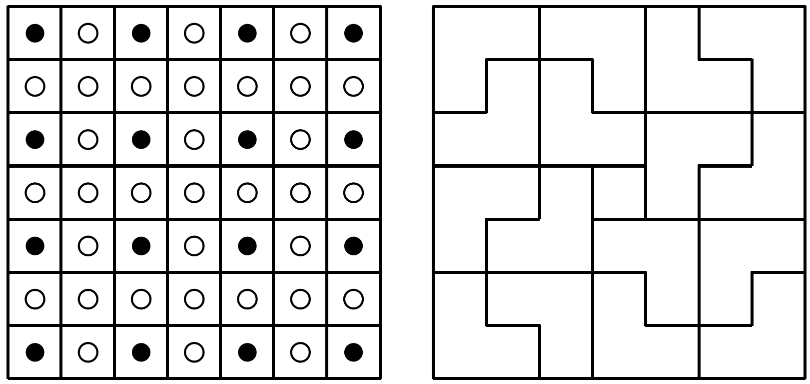
1. **Solution by Howard Halim**
   Suppose after n moves, every boy has a different number of Pokemon cards. For $1 \leq k \leq 10$, let the kth boy give away cards in m_k moves and receives cards in $n - m_k$ moves. Then he will end up with $100 + (n - m_k) - 9m_k = 100 + n - 10m_k$ cards. Each boy will end up with a different number of cards if and only if m_k are distinct. Now $n = m_1 + m_2 + \cdots + m_{10}$. Since each m_k is a non-negative integer, the minimum value of n is $0 + 1 + 2 + \cdots + 9 = 45$.

2. **Solution by Elnaz Hessami-Pilehrood**
   It is possible. Take the board to be the standard chessboard. Put the numbers from 1 to 32 in any order into the white squares, each number appearing exactly once. Put the same number, arbitrarily chosen from 1 to 32, in each of the black square. For any two dominoes, the numbers in the white squares which they cover are different while the numbers in the black squares which they cover are the same. Hence the sums are different.

3. Let A, B, C, D, E and F be the orthocentres in cyclic order. Let B be the orthocentre of triangle PQR as shown in the diagram. Then QR is parallel to CE and RP is parallel to EA. B is the point of intersection of the altitudes PX and QY, which pass through C and A respectively. Since $\angle BCE = 90° = \angle BAE$, the circumcircle of triangle ACE passes through B. Similarly, it passes through D and F.



4. If the initial arrangement is as shown in the diagram below on the right, Alex cannot eat any of the 16 dark chocolates. Since there are 33 white chocolates and he can eat 2 pieces at a time, the most he can eat is 32 pieces. He can guarantee to eat 32 pieces because the box contains 16 disjoint V-trominoes, as shown in the diagram below on the right. The 3 pieces in each V-tromino are adjacent to one another, and by the Pigeonhole Principle, 2 of the pieces must be of the same kind. Hence Alex can eat 2 pieces from each of the 16 V-tromino.

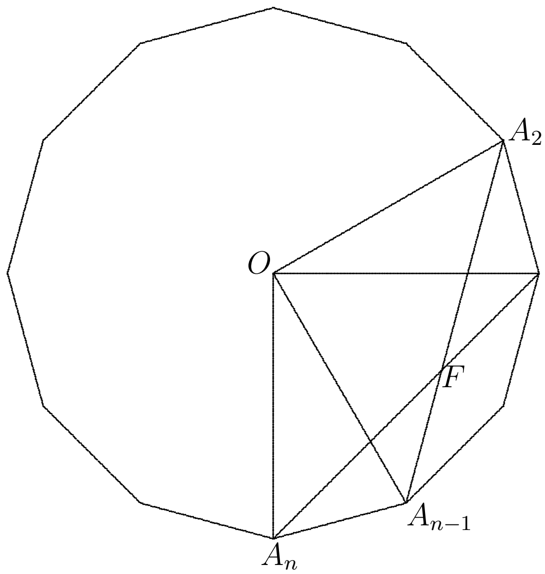


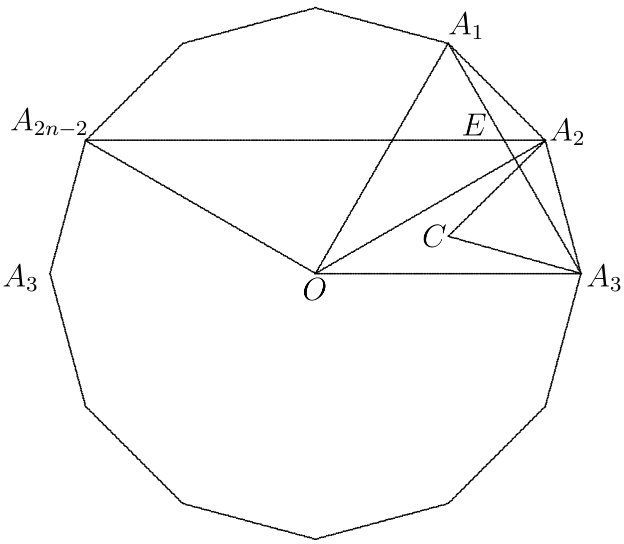
5. **Solution by Olga Ivrii-Zaitseva**
Let the three numbers be x, y and z. Since their pairwise products are all positive, they are either all positive or all negative. Since their pairwise sums are also positive, they are all positive. We may assume that $x > y > z > 0$. Let one piece of paper contain the numbers $a > b > c > 0$ and the other piece contain the numbers $d > e > f > 0$. We first assume that a, b and c are the pairwise sums. Then $x + y = a$, $x + z = b$ and $y + z = c$. Hence $x = a + b - c$, $y = c + a - b$ and $z = b + c - a$. If xy, xz and yz do not agree with d, e and f, then the piece containing a, b and c records the pairwise products. We must not jump to conclusion that if they agree, then the piece containing a. b and c records the pairwise sums. We must still rule out the possibility that d, e and f are the pairwise sums of three other numbers $u > v > w > 0$. We will then have

$$\begin{array}{lll} x + y = a = uv & x + z = b = uw & y + z = c = vw \\ xy = d = u + v & xz = e = u + w & yz = f = v + w \end{array}$$

Then $y - z = a - b = u(v - w) = u(d - e) = ux(y - z)$, so that $ux = 1$. Similarly, $x - y = b - c = w(u - v) = w(e - f) = wz(x - y)$, so that $wz = 1$ also. However, $\frac{1}{x} = u > w = \frac{1}{z}$ implies that $z > x$, which is a contradiction. It follows that ambiguity cannot arise, and we can determine which piece of paper records the pairwise sums.

6. **Solution by Victor Rong**
We solve this problem in three steps.
Step 1. (See diagram below on the left.)
Triangles OA_2A_{n-1} and OA_3A_n are congruent. Hence $\angle OA_2F = \angle OA_3F$, so that OA_2A_3F is cyclic. If C is the circumcentre, then $\angle FCA_3 = 2\angle A_{n-1}A_2A_3 = \angle A_{n-1}OA_3 = (n-4)\angle A_2OA_3$. Also, $2\angle OA_2A_{n-1} = 180° - \angle A_2OA_{n-1} = 3\angle A_2OA_3$.





Step 2. (See diagram above on the right.)
Note that EA_2A_3C is cyclic since

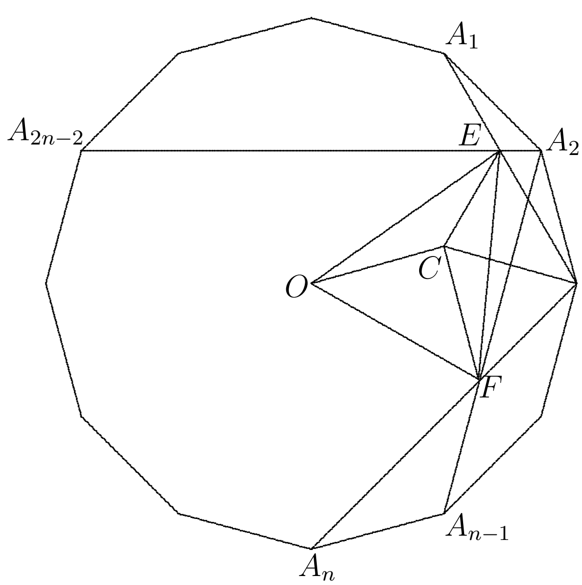
$$\begin{aligned}\angle A_2EA_3 &= \angle A_{2n-2}A_2A_1 + \angle A_3A_1A_2 \\ &= \frac{1}{2}(\angle A_{2n-2}OA_1 + \angle A_3OA_2) \\ &= \frac{1}{2}(3\angle A_3OA_2 + \angle A_3OA_2) \\ &= 2\angle A_3OA_2 \\ &= \angle A_3CA_2.\end{aligned}$$

Hence $\angle A_3CE = 180^\circ - \angle A_{n-2}A_2A_3 = \angle A_{2n-2}A_nA_3 = \frac{1}{2}\angle A_{2n-2}OA_3 = \frac{5}{2}\angle A_2OA_3$.

Step 3. (See diagram below.)
We have $\angle OCF = 2\angle OA_2A_{n-1} = 3\angle A_2OA_3$ by Step 1. By Steps 1 and 2, we have

$$\angle FCE = \angle FCA_3 + \angle A_3CE = (n-4)\angle A_2OA_3 + \frac{5}{2}\angle A_2OA_3 = \frac{1}{2}(2n-3)\angle A_2OA_3.$$

Hence $\angle OCE = 360^\circ - \angle OCF - \angle FCE = (2n-3-\frac{1}{2}(2n-3))\angle A_2OA_3 = \angle FCE$. It follows that $OE = FE$.



7. **Solution by Central Jury**

(a) Let the choices be 0 and 1. A student is said to be odd if the sum of her choices to the first eleven questions is odd. For any ten questions and any of the $2^{10} = 1024$ possible choices, there is a student who makes those choices. Every student can make choices for the remaining 10 questions so that she becomes an odd student. Suppose there exist two students who make different choices for each question. In particular, they will make different choices for the first eleven questions. Then the sum of their choices for these questions will be 11, which means that one of them is not odd. This is a contradiction.

(b) Let the choices be 0, 1, 2, $\ldots$, 11. For $1 \leq k \leq 11$, there is a student whose choice for each of the last ten questions is k. These eleven students will be called the Committee. For each of the first ten questions, there is a choice which is not made by any member of the Committee. On the other hand, there is a student, Charlie, who makes those choices for the first ten questions. For the last ten questions, Charlie makes at most ten different choices. Hence there is a member of the Committee whose choices differ from Charlie's for every question.