International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2016.**

1. Do there exist five positive integers such that their ten pairwise sums end in different digits?

2. Four points on a line plus a fifth point not on the line determine six triangles. At most how many of them can be isosceles?

3. On a circle are 100 points labelled with the positive integers 1 to 100 in some order.

   (a) Prove that these points can be joined in pairs by 50 non-intersecting chords such that the sum of the labels of the two endpoints of each chord is odd.

   (b) Is it always possible to join these points in pairs by 50 non-intersecting chords such that the sum of the labels of the two endpoints of each chord is even?

4. $ABCD$ is a parallelogram. K is a point such that $AK = BD$ and M is the midpoint of CK. Prove that $\angle BMDC = 90°$.

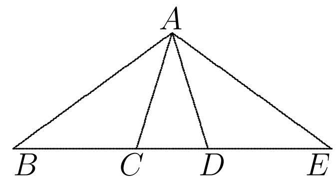
5. One hundred bear-cubs have 1, 2, . . . , 2^{99} berries respectively. A fox chooses two bear-cubs and divide their berries equally between them. If a berry is left over, the fox eats it. What is the least number of berries the fox can leave for the bear-cubs?

Note: The problems are worth 4, 4, 2+2, 5 and 5 points respectively.

Solution to Junior O-Level Fall 2016

1. Consider the sum of these ten sums. It must be even since each of the five original numbers appears four times. However, the sum of ten numbers ending in different digits must be odd. This is a contradiction. Hence no such five positive integers exist.

2. In the diagram below, $\angle ABE = \angle BAC = \angle CAD = \angle DAE = \angle AEB = 36°$. Then we have $BD = AB = AE = EC$ and $BC = CA = AD = DE$. Hence all of ABC, ACD, ADE, EAC, BAD and ABE are isosceles.

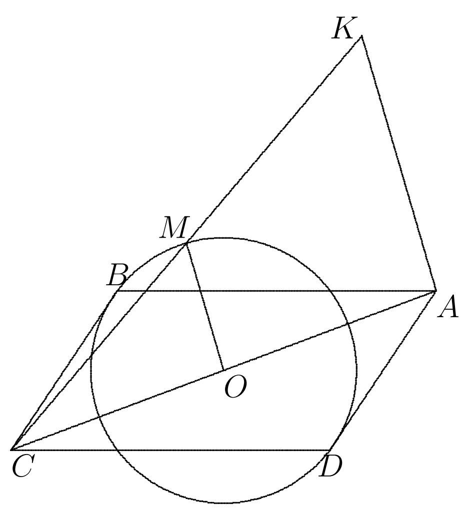


3. Since only parity matters, we may replace all even labels by 0s and all odd labels by 1s. We consider the general case where there are $2n$ points for some positive integer n, half of them labelled 0 and the other half labelled 1.

   (a) We use induction on n. The basis $n = 1$ is trivial. Suppose the result is true for some $n \geq 1$. Consider $2(n+1)$ points. A point labelled 0 must be adjacent to a point labelled 1. Join them by a chord which cannot intersect any other chord. We can apply the induction hypothesis to the remaining $2n$ points.

   (b) Let the 0s and 1s be arranged alternately along the circle. Then we can only join two points with the same label. However, this will leave an odd number of other points on each side of this chord, and some other chord must intersect it.

4. Let O be the centre of $ABCD$. Now K lies on a circle with centre A and radius $AK = BD$. Then M lies on a circle with centre O and radius $\frac{1}{2}BD$. Hence BD is a diameter of this circle, and $\angle BMD = 90°$.



5. Since each bear-cub starts with at least 1 berry, it will end up with at least 1 berry. Thus the least number of berries the fox can leave for the bear-cubs is 1 each. We show more generally that if there are n bear-cubs with 1, 2, ..., 2^{n-1} berries initially, then the fox can accomplish this task. Arrange the bear-cubs in ascending order of the number of berries, so that we start with $(1, 2, \dots, 2^{n-1})$. All sharings are between adjacent bear-cubs, so that their order does not change. We claim that for all $n \geq 1$, (1) $(1, 2, \dots, 2^{n-1})$ can be converted to $(1, 1, \dots, 1, 2^{n-1})$, and (2) $(1, 1, \dots, 1, 2^{n-1})$ can be converted to $(1, 1, \dots, 1)$. The basis $n = 2$ is trivial. Assume that (1) and (2) holds for some $n \geq 2$. Consider $(1, 2, \dots, 2^n)$. Applying the induction hypothesis to the first n terms, we can convert $(1, 2, \dots, 2^n)$ to $(1, 1, \dots, 1, 2^n)$ by (1) and then (2). A sharing between the last two bear-cubs produces $(1, 1, \dots, 1, 2^{n-1}, 2^{n-1})$. Applying the induction hypothesis to the first n terms, we have $(1, 1, \dots, 1, 2^{n-1})$ by (2). Applying the induction hypothesis to the last n terms, we have $(1, 1, \dots, 1)$ by (2). This completes the inductive argument.