International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2016.**

1. Each of ten boys has 100 pokemon cards. In each move, one of the boys gives one card to each of the other boys. What is the minimal number of moves before each boy has a different number of cards?

2. There are 64 positive integers in the squares of an 8×8 board. Whenever the board is covered by 32 dominoes, the sum of the two integers covered by each domino is unique. Is it possible that the largest integer on the board does not exceed 32?

3. The circumcentre O of a quadrilateral $ABCD$ does not lie on either diagonal. The circumcircle of AOC passes through the midpoint of BD. Prove that the circumcircle of BOD passes through the midpoint of AC.

4. The 2016 pairwise sums of 64 numbers are recorded on one piece of paper. They are distinct and positive. The 2016 pairwise products of the same 64 numbers are recorded on another piece of paper. They are also distinct and positive. Later, it is forgotten which piece is which. Is it still possible to determine which piece is which?

5. Is it possible to cut a 1×1 square into two pieces which can cover a disk of diameter greater than 1?

6. Alice chooses a polynomial $P(x)$ with integer coefficients. In each move, Bob gives Alice an integer a, and Alice tells him the number of different integer solutions of the equation $P(x) = a$. Bob may not give Alice the same number twice. Determine the minimal number of moves for Bob to make Alice tell him a number that she has told him before, regardless of the polynomial chosen by Alice.

7. A finite number of frogs are placed on distinct integer points on the real line. At each move, a single frog jumps by 1 to the right provided that the new location is unoccupied. Altogether, the frogs make n moves, and this can be done in m ways. Prove that if they jump by 1 to the left instead of to right, they can still make n moves in m ways.

Note: The problems are worth 5, 5, 7, 8, 9, 9 and 12 points respectively.

Solution to Senior A-Level Fall 2016

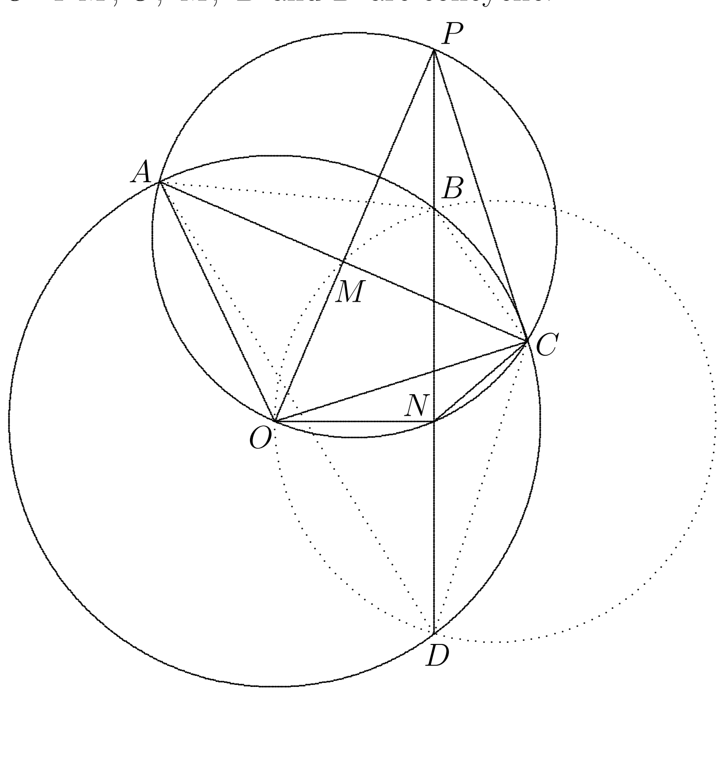
1. **Solution by Howard Halim**
   Suppose after n moves, every boy has a different number of Pokemon cards. For $1 \le k \le 10$, let the kth boy give away cards in m_k moves and receives cards in $n - m_k$ moves. Then he will end up with $100 + (n - m_k) - 9m_k = 100 + n - 10m_k$ cards. Each boy will end up with a different number of cards if and only if m_k are distinct. Now $n = m_1 + m_2 + \cdots + m_{10}$. Since each m_k is a non-negative integer, the minimum value of n is $0 + 1 + 2 + \cdots + 9 = 45$.

2. **Solution by Elnaz Hessami-Pilehrood**
   It is possible. Take the board to be the standard chessboard. Put the numbers from 1 to 32 in any order into the white squares, each number appearing exactly once. Put the same number, arbitrarily chosen from 1 to 32, in each of the black square. For any two dominoes, the numbers in the white squares which they cover are different while the numbers in the black squares which they cover are the same. Hence the sums are different.

3. **Solution by Pierre Haas**
   Let the extensions of OM and DB meet at P. Then

$$\begin{aligned}\angle OCN &= 180^\circ - \angle ONC - \angle CON \\ &= \angle OAC - \angle CON \\ &= 90^\circ - \angle MOC - \angle CON \\ &= 90^\circ - \angle MON \\ &= \angle OPN.\end{aligned}$$

   Hence P lies on the circle passing through A, O, N and C. Moreover, OP is a diameter of this circle. Since $\angle OCP = 90^\circ$, CP is a tangent to the circumcircle of $ABCD$. It follows that $CP^2 = PB \cdot PD$. Also, triangle POC is similar to triangle PCM, so that $CP^2 = PO \cdot PM$. From $PB \cdot PD = PO \cdot PM$, O, M, B and D are concyclic.



4. **Solution by Olga Ivrii-Zaitseva**

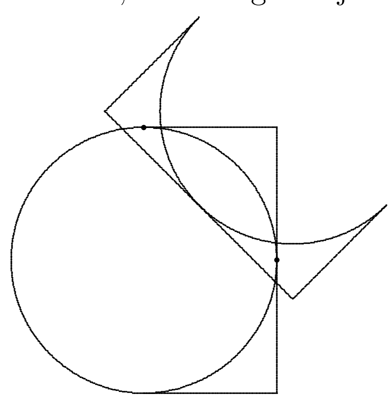
Let one piece of paper contain the numbers $a_1 > a_2 > \cdots > a_{2016} > 0$ and the other piece contain the numbers $b_1 > b_2 > \cdots > b_{2016} > 0$. Since the pairwise products of the 64 numbers are all positive, they are either all positive or all negative. Since their pairwise sums are also positive, they are all positive. We first show that ambiguity cannot arise. Suppose $x_1 > x_2 > \cdots > x_{64} > 0$ are such that their pairwise sums are the as and their pairwise products are the bs, while $y_1 > y_2 > \cdots > y_{64} > 0$ are such that their pairwise sums are the bs and their pairwise products are the as. Then $x_1 + x_2 = a_1 = y_1y_2$, $x_1 + x_3 = a_2 = y_1y_3$, $y_1 + y_2 = b_1 = x_1x_2$ and $y_1 + y_3 = b_2 = x_1x_3$. Hence

$$a_1 - a_2 = x_2 - x_3 = y_1(y_2 - y_3) = y_1(b_1 - b_2) = x_1y_1(x_2 - x_3) = x_1y_1(a_1 - a_2).$$

It follows that $x_1y_1 = 1$. Similarly, we can show that $x_{64}y_{64} = 1$. This is a contradiction since $x_1y_1 > x_{64}y_{64}$. Let $x_1 > x_2 > \cdots > x_{64} > 0$ be the numbers. We assume that the as are the pairwise sums. Then $x_1 + x_2 = a_1$, $x_1 + x_3 = a_2$, $x_1x_2 = b_1$ and $x_1x_3 = b_2$. Then $a_1 - a_2 = x_2 - x_3$ while $b_1 - b_2 = x_1(x_2 - x_3)$. Hence $x_1 = \frac{b_1 - b_2}{a_1 - a_2}$. Now we check whether $a_1 - x_1 = \frac{b_1}{x_1}$ and $a_2 - x_1 = \frac{b_2}{x_1}$. If so, our assumption is correct. Otherwise, the as are the pairwise products.

5. **Solution by Edmund Kong**

A unit circle is inscribed in a unit square. The left half of the square outside the circle is cut off and placed as shown in the diagram, covering the points of tangency on the top and right edges of the square. If we increase the radius of the circle slightly, keeping it tangent to the bottom and the left edge, the new area beyond the square can be covered by the piece. Also, this piece is hanging together by a thread, but a slight adjustment can make it connected.



6. **Solution by Victor Ivrii**

Bob can guarantee a win in 4 moves but not in 3. In 3 moves, whatever numbers Bob gives, Alice can always answer 0, 1 and 2. Let Bob's numbers a, b and c in that order. Alice can show Bob that her polynomial is $P(x) = (c - b)x^{2n} + b$ for some suitably chosen positive integer n. When $P(x) = c$, $(c - b)(x^{2n} - 1) = 0$. Since $c \neq b$, $x^{2n} = 1$ and there are indeed two distinct integer roots, namely, ± 1. When $P(x) = b$, $(c - b)x^{2n} = 0$. Since $c \neq b$, $x^{2n} = 0$ and $x = 0$ is the only root. Finally, when $P(x) = a$, the only roots are $\sqrt[2n]{\frac{a-b}{c-b}}$. If $\frac{a-b}{c-b} < 1$, the roots are either non-real or non-integral. Since $a \neq c$, $\frac{a-b}{c-b} \neq 1$. Suppose $\frac{a-b}{c-b} > 1$. Since the positive integer n is arbitary, it can always be chosen so that none of the roots are integral.

We now prove that 4 moves are sufficient. Note that for integers x and y, $|P(x) - P(y)| = 1$ implies $|x-y| = 1$ since $P(x)-P(y)$ is divisible by $x-y$. Bob starts by calling 0. We consider four cases.

Case 1. Alice's answer is at least 3.

This means that $P(x) = 0$ has at least 3 distinct integral roots. Now every root of $P(x) = \pm 1$ must differ by exactly 1 from every root of $P(x) = 0$. Clearly, $P(x) = \pm 1$ cannot have integral roots. By calling 1 and -1, Bob wins in 3 moves.

Case 2. Alice's answer is 2.

Bob can win as in Case 1 unless the two roots of $P(x) = 0$ are integers differing by 2. Suppose Bob does not win by calling 1 and -1. We may assume by symmetry that $P(x) = 1$ has no integral roots and $P(x) = -1$ has one. Now $P(x) = -2$ can have at most 2 distinct integral roots, and Bob wins by calling -2.

Case 3. Alice's answer is 1.

Then each of $P(x) = 1$ and $P(x) = -1$ has at most 2 distinct integral roots. If Bob does not win after calling 1 and -1, we may assume by symmetry that $P(x) = 1$ has no integral roots while $P(x) = -1$ has two. Then $P(x) = -2$ has at most one integral root, and Bob wins by calling -2.

Case 4. Alice's answer is 0.

Bob calls -1. We consider four subcases.

Subcase 4(a). Alice's answer is 0.

Then Bob has won already.

Subcase 4(b). Alice's answer is 1.

Bob calls -2. Since $P(x) = -2$ has at most two distinct integral roots, Bob wins by calling -2 unless it has exactly two such roots. Now Bob wins by calling -3.

Subcase 4(c). Alice's answer is 2.

Then $P(x) = -2$ has at most one integral root, and Bob wins by calling -2 unless it has exactly one integral root. Now Bob wins by calling -3.

Subcase 4(d). Alice's answer is at least 3.

Bob wins by calling -2 since $P(x) = -2$ cannot have any integral roots.

7. **Solution by Central Jury**

Let $S = \{a_1, a_2, \ldots, a_n\}$, where each a_k, $1 \leq k \leq n$, is either ℓ or r. It denotes a sequence of jumps where the kth jump is to the left if $a_n = \ell$ and to the right if $a_k = r$. Let $f(S)$ be the number of possible ways of carrying out S from the initial configuration of frogs. We claim that $f(\ell\ell\ldots\ell) = f(rr\ldots r)$. Note that $f(S\ell) = f(Sr)$ for any S. Clearly, we have equality up to the last jump, and the number of possible last jumps depends only on the number of groups of adjacent frogs at that point. We also have $f(S\ell rS') = f(Sr\ell S')$ for any S and S'. Again, we have equality up to the completion of S. Consider $S\ell rS'$. If the two switched jumps are made by different frogs, we can just switch their order and continues. If they are made by the same frog but not the leftmost one, we can replace them with jumps r and ℓ by the frog immediately to its left, with at least one space in between. Finally, if both jumps are made by the leftmost frog, we can replace them with jumps r and ℓ by the rightmost frog. These transformations allow us to conclude that $f(S)$ depends only on the length of S.