International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2016.**

1. Two parabolas are the graphs of quadratic trinomials with leading coefficients p and q respectively. They have different vertices which lie on each other. What are the possible values of $p+q$?

2. Of the triangles determined by 100 points on a line plus an extra point not on the line, at most how many of them can be isosceles?

3. One hundred bear-cubs have 1, 2, $\ldots$, 2^{99} berries respectively. A fox chooses two bear-cubs and divide their berries equally between them. If a berry is left over, the fox eats it. What is the greatest number of berries the fox can eat?

4. Let n be a positive integer. An n-omino is a figure consisting of n unit squares joined edge to edge. A 100-omino can be dissected into two congruent 50-ominoes as well as 25 congruent tetrominoes. Is it always possible to dissect it into 50 dominoes?

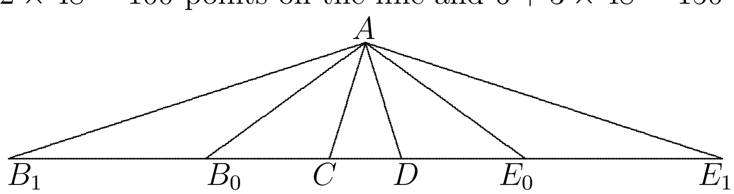
5. Prove that in a right triangle, the altitude to the hypotenuse passes through the orthocentre of the triangle determined by the points of tangency of the incircle with the right triangle.

Note: The problems are worth 4, 5, 5, 5 and 5 points respectively.

Solution to Senior O-Level Fall 2016

1. **Solution by Olga Irvii-Zaitseva.**
Translation of a parabola does not affect the coefficient of its quadratic term. If we translate the two parabolas as a single entity, their vertices will still lie on each other. We translate them so that one of them has vertex (0,0). Its equation will be $y = px^2$. Let (a, b) be the vertex of the other parabola after translation. Its equation will be $y = q(x-a)^2 + b$. Since the two vertices do not coincide, $a \neq 0$. Putting (0,0) into the second equation and (a, b) into the first, we have $0 = qa^2 + b$ and $b = pa^2$. Addition yields $0 = a^2(p+q)$. Since $a \neq 0$, we must have $p + q = 0$.

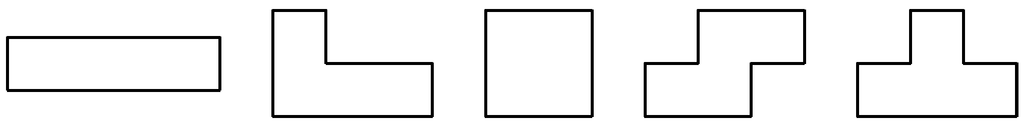
2. **Solution by Steven Chow.**
In the diagram below, $\angle AB_0E_0 = \angle B_0AC = \angle CAD = \angle DAE_0 = \angle AE_0B_0 = 36°$. Then we have $B_0D = AB_0 = AE_0 = E_0C$ and $B_0C = CA = AD = DE_0$. Hence all of AB_0C, ACD, ADE_0, E_0AC, B_0AD and AB_0E_0 are isosceles. We now add B_1 and E_1 so that $B_1B_0 = B_0A = AE_0 = E_0E_1$. Then we have three more isosceles triangles, namely, AB_1B_0, AE_0E_1 and AB_1E_1. Continuing this way, we add points B_n and E_n up to $n = 48$. This gives us $4 + 2 \times 48 = 100$ points on the line and $6 + 3 \times 48 = 150$ isosceles triangles.



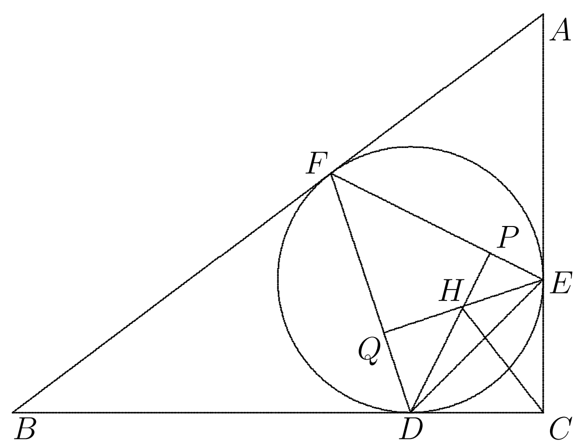
We now prove that 150 is indeed the maximum. Call the vertex the pivot of an isosceles triangle if it is the common vertex of the two equal sides. There are two kinds of isosceles triangles, those whose pivot is the point A outside the line, and those whose pivot is on the line. Since only 100 segments has A as an endpoint, the number of isosceles triangles of the first kind is at most 50. Each point P on the line is a non-pivot of at most one isosceles triangle of the second kind, since the pivot must lie on the perpendicular bisector of AP, which intersects the line at a unique point. It follows that the number of such triangles is at most 100.

3. Since each bear-cub starts with at least 1 berry, it will end up with at least 1 berry. Thus the least number of berries the fox can leave for the bear-cubs is 1 each. We show more generally that if there are n bear-cubs with 1, 2, ..., 2^{n-1} berries initially, then the fox can accomplish this task. Arrange the bear-cubs in ascending order of the number of berries, so that we start with $(1, 2, \ldots, 2^{n-1})$. All sharings are between adjacent bear-cubs, so that their order does not change. We claim that for all $n \geq 1$, (1) $(1, 2, \ldots, 2^{n-1})$ can be converted to $(1, 1, \ldots, 1, 2^{n-1})$, and (2) $(1, 1, \ldots, 1, 2^{n-1})$ can be converted to $(1, 1, \ldots, 1)$. The basis $n = 2$ is trivial. Assume that (1) and (2) holds for some $n \geq 2$. Consider $(1, 2, \ldots, 2^n)$. Applying the induction hypothesis to the first n terms, we can convert $(1, 2, \ldots, 2^n)$ to $(1, 1, \ldots, 1, 2^n)$ by (1) and then (2). A sharing between the last two bear-cubs produces $(1, 1, \ldots, 1, 2^{n-1}, 2^{n-1})$. Applying the induction hypothesis to the first n terms, we have $(1, 1, \ldots, 1, 2^{n-1})$ by (2). Applying the induction hypothesis to the last n terms, we have $(1, 1, \ldots, 1)$ by (2). This completes the inductive argument. Thus the greatest number of berries the fox can eat is $1 + 2 + \cdots + 2^{n-1} - n = 2^n - n - 1$. For $n = 100$, this number is $2^{100} - 101$.

4. There are only five different tetrominoes, as shown in the diagram below. Each of the first four can be dissected into two dominoes. Assume to the contrary that the desired task is not possible. Then the 100-domino must be dissected into 25 copies of the last tetromino. When placed on a standard chessboard, this tetromino covers three unit squares of one colour and one of the other colour. Hence 25 copies of it will cover an odd number of white squares and an odd number of black squares. Now the 100-omino can also be dissected into two congruent 50-ominoes. If black squares in one copy correspond to white squares in the other copy, then the 100-omino must cover 50 white squares and 50 black squares. If black squares in one copy correspond to black squares in the other copy, the 100-omino must cover an even number of white squares and an even number of black squares. Either way, we have a contradiction. Thus the desired task is always possible.



5. **Solution by Olga Irvii-Zaitseva.**
Let the triangle be ABC with a right angle at C. Let the incircle touch BC, CA and AB at D, E and F respectively. Let the altitudes DP and EQ of DEF intersects at the orthocentre H of DEF.



Note that $\angle BFD + \angle AFE = \frac{1}{2}(2 \times 180° - \angle ABC - \angle BAC) = 135°$. Hence $\angle DFE = 45°$ so that $\angle PDF = 45° = \angle FEQ$. Since $FPHQ$ is cyclic, $\angle DHE = \angle PHQ = 135°$, so that $\angle DHE + \frac{1}{2}\angle DCE = 180°$. It follows that H lies on the circle with centre C and passing through D and E. Now $\angle EHC = \angle HEC = 135° - \angle PEA = 135° - \angle PFA = \angle BFD$. Hence $\angle HCA = \angle FBD$, so that CH is perpendicular to AB.